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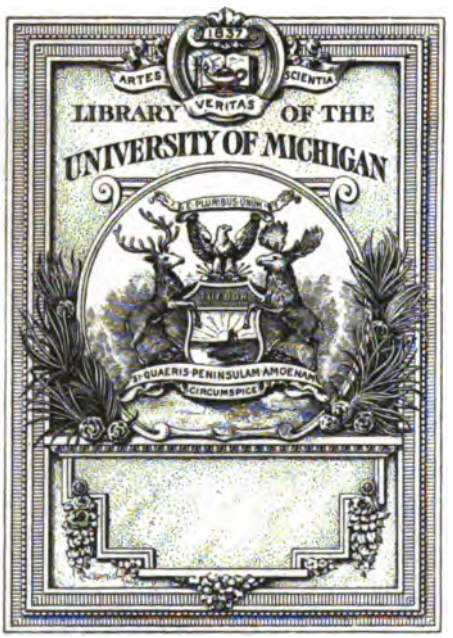
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THE

MATHEMATICAL MONTHLY.

EDITED BY

J. D. RUNKLE, A.M.,

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CONTENTS OF VOL. III.

No. I. OCTOBER, 1860.

	Page
Prize Problems for Students	1
Note on some Properties of Square Numbers. By Prof. D. W. Hoyt	2
Note on the "Horizontal Thrust of Embankments." By Prof. F. W. Bardwell	6
Note on the Dependence of Equations. By John Borden	7
Note on Numerical Equations. By Prof. A. Vallas	10
Report of the Judges upon the Solutions of the Prize Problems in No. IX., Vol. II.	18
Report of the Judges upon the Solutions of the Prize Problems in No. X., Vol. II.	22
Discussion of the Equations which determine the Position of a Comet or other Planetary Body from three Observations. By G. W. Hill	26
Mathematical Monthly Notices	30

No. II. NOVEMBER, 1860.

	Page
Prize Problems for Students	33
Remarks on the "Problem of the Lights." By Prof. W. H. Parker	34
Note on the "Problem of the Lights." By Hugh Godfray, Esq.	38
An Easy Mode of Approximating to the Time of Vibration in a Circular Arc. By Prof. L. R. Gibbes	40
Report of the Judges upon the Solutions of the Prize Problems in No. XI., Vol. II.	46
Notes and Queries	51
Note on the Greatest Common Measure. By E. Hunt, Esq.	51
Notes from Todhunter's Algebra	52
Exposition of the Process of Mathematical Development. By John Paterson	55
Mathematical Monthly Notices	61
Editorial Items	64

No. III. DECEMBER, 1860.

Prize Problems for Students . . . 65
Notes and Queries. — Perfect Squares — Problem — Note on Prize Solution of Problem III., p. 25 — Correction of the Method of Computing Six Per Cent Interest for Days — Solution of Problem V., No. IX., Vol. II. — Note on Powers of Binomials — The Arithmetic Mean of any Number of Positive Quantities is Greater than the Geometric — Note on the Successive Derivatives of $\tan\varphi$ 66–70
The Earth considered as a Spheroid of Revolution. — Geodetic Formulas. By Dr. R. C. Matthewson . . 71
On Spherical Analysis. By George Eastwood 78
A Treatise on Determinants. By James Edward Oliver 86
Mathematical Monthly Notices . . 90

No. IV. JANUARY, 1861.

Prize Problems for Students . . . 97
Notes and Queries. — To Express the Sines and Cosines of the Sum and Difference of Two Angles in Terms of the Sines and Cosines of the Angles themselves. — In a Spherical Triangle, to express the Cosine of an Angle in Terms of the Sines and Cosines of the Sides. — Note on Napier's Rules of Circular Parts 98
Report of the Judges upon the Solutions of the Prize Problems in No. XII., Vol. II. 104
Exposition of the Process of Mathematical Development. By John Paterson 108
Notes on Probabilities. By Simon Newcomb 119
Mathematical Monthly Notices . . 125

No. V. FEBRUARY, 1861.

Prize Problems for Students . . . 129
Report of the Judges upon the Solutions of the Prize Problems in No. I., Vol. III. 130
Descriptive Geometry of One Plane. By William Watson 133
Notes and Queries 149
A General Method of Finding Tangents to Algebraic Curves. By Walter Holliday 149
Squares of Numbers ending in 5, 25, and 75. By J. F. Roberson . . . 150
Solution of Problem V., No. X., Vol. II. By Asher B. Evans . . . 151
On an Approximate and Graphical Rectification of the Circle. By A. S. Herschel 152
Mathematical Monthly Notices . . 155
Editorial Items 160

No. VI. MARCH, 1861.

Prize Problems for Students . . . 161
Award of the Prizes for Solutions of Problems in No. II., Vol. III. . . . 162
First Prize Essay. — On the Conformation of the Earth. By G. W. Hill . . 166
Notes and Queries. — A Logical Outline of Arithmetic. By Prof. E. Brooks . 182
Notes. By Prof. M. C. Stevens . . 184
Notes on Geometry. By Pike Powers 185
Problem. By David Trowbridge . 187
Magic Table. By Artemas Martin 189
Note. By Prof. D. W. Hoyt . . 189
Mathematical Monthly Notices . . 190
Editorial Items 192

No. VII. APRIL, 1861.

Prize Problems for Students . . . 193
Award of the Prizes for Solutions of Problems in No. III., Vol. III. . . . 194
Solution of Numerical Equations of the Third Degree. By Prof. J. C. Porter 198
Motion on an Inclined Plane. By Thomas Sherwin 206
The Theory of the Gyroscope. By Theodore G. Ellis 209
A Complete Catalogue of the Writings of Sir John Herschel. Prepared by the Author 220
Editorial Items 228

No. VIII. MAY, 1861.

Prize Problems for Students . . . 229
Award of the Prizes for Solutions of Problems in No. IV., Vol. III. . . . 230
On the Geometrical Construction of Certain Curves by Points. By Prof. H. A. Newton 235
Second Prize Essay. — On Central and Disturbing Forces. By David Trowbridge 245
Note on a Question in Interest. By Prof. J. C. Porter 254
Note on the General Theory of Equations. By Prof. E. W. Evans . . 256
Note on Trigonometrical Formulas. By Prof. R. M. Moore 258
Mathematical Monthly Notices . . 259

No. IX. JUNE, 1861.

Prize Problems for Students . . . 261
To change a Series into a Continued Fraction. By Asaph Hall . . 262
On the Geometrical Construction of Certain Curves by Points. By Prof. H. A. Newton 268
Award of the Prizes for Solutions of Problems in No. V., Vol. III. . . . 280
Demonstration of the Law of Equilibrium in the Lever. By Dr. Theo. Strong 283
Mathematical Monthly Notices . . 286
Editorial Items 292

No. X. JULY, 1861.

Third Prize Essay. — The Method of Projections. By Arthur W. Wright 293
Mathematical Notes. By Matthew Collins 306
The Pendulum. By Thomas Sherwin 312
The Polytechnic School at Paris. By William Watson 320

No. XI. AUGUST, 1861.

On the Diophantine Analysis. By Rev. A. D. Wheeler 325
Notes on the Theory of Probability. — Insurance. By Simon Newcomb . 343
Award of the Prizes for Solutions of Problems in No. VI., Vol. III. . . . 349
The École des Ponts et Chaussées at Paris 351
Mathematical Monthly Notices . . 354

No. XII. SEPTEMBER, 1861.

On the Results of Accelerated Velocity. By JOHN PATERSON 357
Fourth Prize Essay. — Spherical Conics. By C. M. WOODWARD . . . 361
Equation and Construction of the Hyperbola. By JOHN WARNER . . 375
Convenient Formulæ for Interpolation. By WILLIAM FERREL 377

ALPHABETICAL INDEX 385

INDEX OF PROPER NAMES.

Names of Contributors are in Small Capitals.

Adams, 225
Adamson, 223
Airy, 96, 225
Allaire, 312
Apollonius, 287
Arago, 221
Archimedes, 287
Argelander, 225

Babbage, 220
Bache, 190
Bacon, 287
Bagley, 349
Baily, 222, 223
BARDWELL, 6
BARKER, 22, 130
Barrow, 286
BARTLETT, 26, 50, 107, 132
Beattie, 193
Beebe, 93
Berthollet, 321
Bessel, 222
Biela, 222
Bigelow, 93
Billy, 305
Binney, 93
Biot, 259
Bishop, 225
Bonaparte, 322
Bond, George P., 356
Bond, R. F., 64
Booth, 226
BORDEN, 7
Borelli, 288
Bosworth, 292
Boudrot, 305
Bourdon, 40
Bowditch, 222
Brewer, 190
Brewster, 220
Brianchon, 294, 305
Brinkley, 226
Brioschi, 89
Brisbane, 222
Brooks, C. H., 191
BROOKS, E., 184
Brunacci, 354
Brünnow, 64
Buck, 287
Buckley, 192
Budan, 10

Cabot, 93
Canfield, 292
Carnot, 156, 294, 305
Carpenter, 190
Cartesius, 287, 288
Cauchy, 354
Cavalieri, 260
Cayley, 89, 160
Cecil, 286
Chasles, 156, 235, 268, 276, 305
Chauvenet, 95
Christie, 224
Church, 228, 289
Clairaut, 175
Clark, 228
Clerihews, 225
COTTER, 161
Collins, 283
COLLINS, M., 306
COMFORT, 280
Cooke, 30
Corey, 280
Cotes, 220
Cousinery, 294
Crelle, 268, 305
Curtis, 192

Dalton, 93
Davies, 30, 63
DAVIS, 27
Davison, 18
Daubeny, 222
De Candolle, 190
De Freycinet, 356
De Jonquieres, 268
De La Hire, 293
Delambre, 96
Delaunay, 354
De Morgan, 95, 259
Desargues, 293, 294
Descartes, 260
Dirkson, 354
Dobson, 191
Dodd, 192, 290
Duhamel, 158
Duiong, 30
Dunlop, 222

EASTWOOD, 78
Ecchellesius, 288
Edleston, 259
Elder, 225
Ely, 225
Ellinwood, 192
ELLIS, 209
Emerson, 93
Euler, 55
EVANS, A. B., 1, 18, 21, 24, 46, 104, 106, 152, 162, 194, 196, 223, 349
EVANS, E. W., 161, 258
EVANS, W. H., 280
Everest, 225

Fal de Bruno, 89
Fenwick, 191
Ferguson, 64
FERREL, 160, 377
Ferrers, 61
Flint, 93
Fourier, 10, 11
FRANKENSTEIN, 19
Fraunhofer, 221
Frost, 61

Galileo, 287
Ganot, 128
Gassendi, 287
Gauss, 27, 268, 354
Gear, 292
Gerhardt, 259
Gergonne, 268, 276, 293, 305
Gerono, 305
GIBBES, 40
GIBSON, 46, 130
Gilbert, 287
Gillis, 190
Gookin, 93
Golius, 288
GODFRAY, 88, 94
Greenleaf, 32
Griffin, 292
Grunert, 305

Hachette, 321
HALL, 33, 262
Halley, 223, 288
Hamilton, 224
Hansen, 127, 225
HARELTINE, 18
HENCK, 160
Hencke, 225
Henderson, 228
HENKLS, 66
Henry, 190
Herschel, A. S., 155
Herschel, Caroline L., 225
Herschel, Sir John, 192, 220
Herschel, Sir W., 227
HICKS, 129, 165, 194, 230, 232
HILL, 26, 166, 192
Hind, 225
Hipparchus, 293
Holcomb, 292
HOLLIDAY, 150
Hollister, 230
Horner, 10, 11
Houseau, 225
HOYT, 2, 67, 189, 255
Humboldt, 225
HUNT, 51

Jacobi 354
Jamieson, 221
Johnson, S. W., 190
Johnson, W. W., 192
Joule, 30

Keill, 259
Kemp, 226
Knatchbul, 286
Kneeland, 93
Knight, 222

Lacroix, 220, 354
Lagrange, 160, 321, 354
Lamarle, 356
Laplace, 64
Lardner, 222
Lassell, 225
Le Conte, 190
Lefort, 259
Legendre, 354
Leibnitz, 160, 259
LEMMON, 194, 231
LEONARD, 130
Leverrier, 225
Levy, 190
Lewis, 192
Leybourne, 220
Liouville, 156, 268, 305
Lloyd, 224
Loomis, 190
Lovelace, 225
Lubbock, 225
Lucas, 285
Lyell, 223

Maclaurin, 32
Maedler, 190, 228
Mann, 292
Marcy, 228, 349
Marston, 292
MARTIN, 189
Mason, 191
Masterman, 355
MATTHEWSON, 71
Maury, 64
McMurphy, 228
Mersenne, 287, 288
Merahon, 192
Mitchell, 228
Monge, 156, 294
Montucla, 158, 288
MOORE, 258
Morgan, 64
Murchison, 223
Murdock, 356

Napier, 96, 101

Newcomb, 26, 50, 69, 95, 107, 119, 132, 343
Newton, 10, 11, 160, 259, 288, 293
Newton, H. A., 235, 268
Nicholson, 220
Norton, 356

Ohm, 354
Oliver, 336
Olivier, 156
Ordway, 93
Osborne, 162, 164
Ostrogradsky, 354
Oughtred, 286
Overstone, 225

Parker, 34
Pascal, 293
Paterson, 55, 108, 357
Peacock, 96, 220, 226
Pearson, 222
Peck, 128
Peirce, 70, 89, 127
Perronet, 351
Peters, 64, 356
Philbrick, 93
Phillips, 222
Pierce, 230
Plana, 221
Plücker, 305
Poisson, 354
Poncelet, 156, 276, 294
Pontécoulant, 26
Porter, 199, 256
Powers, 187
Pratt, 93
Proney, 321
Ptolemæus, 293
Puissant, 312

Quetelet, 222, 225, 305

Raworth, 287
Ray, 286
Regnault, 30
Rheticus, 96
Rhoades, 230
Rice, 93
Ritchie, 221
Robertson, 129, 151
Robertson, 223
Roberval 158
Robinson, 90, 126, 160, 189, 254
Rogers, W. A., 356
Rogers, W. B., 93
Ross, 93
Rotch, 93
Routh, 61, 228
Rutter, 190

Sabine, 224
Safford, 26, 50, 107, 132
Salmon, 89, 185, 305
Sarrus, 354
Schiller, 224
Schlömilch, 268
Schubert, 356
Schumacher, 221, 223
Schuyler, 192
Scott, 67
Sechler, 192
Serret, 158
Servois, 293
Sherwin, 206, 312
Sloman, 192, 259
Smith, H. W., 192
Smith, James, 291
Smyth, 159
Smythe, 225
South, 221
Spence, 220
Spottiswoode, 89
Stanley, 280
Stearns, 195
Stern, 268
Stevens, 185
Stevelly, 224
Stewart, 226
Stockwell, 89
Stratford, 222
Strong, 288

Tacquet, 286
Taylor, 82, 91
Tebay, 191
Telman, 130
Terquem, 305
Terrill, 104
Theodosius, 287
Timson, 22
Tobey, 93
Todhunter, 52, 62, 70, 96, 103, 354
Trowbridge, 162, 188, 194, 197, 230, 232, 245, 350
Tyndale, 226

Vallas, 10
Vieta, 260, 288

Wallis, 260
Warren, 155
Waterston, 93
Watson, H. W., 61
Watson, Stephen, 191
Watson, William, 64
Weber, 224
Weisbach, 6
Weisse, 225
Wheeler, A. D., 225
Wheeler, O. B., 18
Whewell, 224, 286
Wilder, 93
Wilkinson, T., 190
Wilkinson, T. T., 191
Willard, C. H., 292
Willard, E. B., 192
Willis, 104
Willoughby, 286
Winebrener, 1, 46, 49, 192, 292
Woodbury, 6
Woodhouse, 96, 354
Woodward, 361
Wright, Arthur W., 293
Wright, Chauncey, 160

Young, 48

THE

MATHEMATICAL MONTHLY.

Vol. III. ... OCTOBER, 1860. ... No. I.

PRIZE PROBLEMS FOR STUDENTS.

I. Let AB and CD be two diameters of a given circle, drawn at right angles to each other; AEB a circular arc described with radius DA or DB. Prove that the area of the lune $AEBC$ equals the area of the triangle ADB.

II. Of all isoperimetrical polygons having the same number of sides, the greatest is that which is equilateral.

III. In any triangle, if P, P', and P'' denote the perpendiculars from the vertices to the opposite sides; p, p', and p'' the portions of the same between their point of intersection and the sides. Prove that

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

— Communicated by J. A. WINEBRENER, Princeton College, N. J.

IV. At what angle must the rudder of a vessel be inclined to the stream, that the effect produced may be a maximum?

V. Two circles, radii r and $r_{,}$, have their centres united by a line d feet long. Required the position upon this line of the centre of a third circle, which shall cut off a square feet from the first circle and b feet from the second. — Communicated by ASHER B. EVANS.

Solutions of these problems must be received by December 1, 1860. (See Editorial Items.)

NOTE* ON SOME PROPERTIES OF SQUARE NUMBERS.

By Prof. D. W. Hoyt, Fairfax, Vt.

If all *differences* of square numbers which produce squares are known, the *sums* of squares which produce squares are also known; for the one is obtained from the other by the transposition of a single term of the equation.

The differences between consecutive squares of the series 0, 1, 4, 9, 16, 25, &c., are 1, 3, 5, 7, 9, &c., a series including all the odd numbers, and consequently the squares of all odd numbers. The square of every odd number is, therefore, equal to the difference between the squares of two numbers differing by unity. 1 is the difference between 0 and 1; the second odd square is 9, which is the difference between 16 and 25; the third is 25, which is the difference between 144 and 169; that is, $1^2 = 1^2 - 0^2$, $3^2 = 5^2 - 4^2$, and $5^2 = 13^2 - 12^2$, from which other equations of the series can readily be obtained, as the common difference of the first members is 2, and the terms of the second members have 4 as a common difference of the second order.

The differences between alternate terms of the series of square numbers are 4, 8, 12, 16, &c., including all numbers divisible by 4; hence the square of every even number is equal to the difference between the squares of two numbers differing by 2. By an inspection of these series, we obtain $2^2 = 2^2 - 0^2$, $4^2 = 5^2 - 3^2$, $6^2 = 10^2 - 8^2$, &c.

The differences between the squares of numbers differing by 3 include all odd numbers divisible by 3, and consequently all those divisible by 9; hence the square of every odd number divisible by 3

* This note was furnished by Prof. Hoyt, in connection with that published on pp. 264, 265, May, 1860, but was then omitted for want of space. — Ed.

is equal to the difference between the squares of two numbers differing by 3. The differences between the squares of numbers differing by 4 include all numbers divisible by 8, and consequently all those divisible by 16, or the squares of those divisible by 4. The use of numbers differing by 5, 6, and 7, as well as 3 and 4, give equations which are multiples of those contained in the first two series; but 8 and 9 introduce new relations of numbers. The differences between the squares of numbers differing by 8, in the series $(-3)^2$, $(-2)^2$, $(-1)^2$, 0^2, 1^2, 2^2, 3^2, 4^2, 5^2, 6^2, &c., are 16, 32, 48, 64, 80, &c., including all numbers divisible by 16; and consequently the squares of all numbers divisible by 4, giving a series of equations whose first members are the same as when the numbers differ by 4. The square of every odd number divisible by 3 is also equal to the difference between the squares of two numbers differing by 9, as well as two others differing by 3.

In general, if m represent any odd number whatever, then the square of every odd number divisible by m will be equal to the difference between the squares of two numbers differing by m. If n represent any even number, the square of every number divisible by n will be equal to the difference between the squares of two numbers differing by n. If m contains a square factor, or if n contains a square factor in addition to the factor 2, there will also be numbers not divisible by m or n whose squares will be equal to the differences between the squares of numbers differing by m or n. The following statement, however, includes all possible cases: Every odd square divisible by m will be equal to the difference between the squares of two numbers differing by m, and every square divisible by $2n$ will be equal to the difference between the squares of two numbers differing by n.

Since $18 = 2 \times 3^2$ and $25 = 5^2$, the use of those numbers as differences will introduce some equations not multiples of any preceding ones.

The same results may be more readily and satisfactorily reached by another process. The general equation is of the form $M^2 = N^2 - P^2$. Let $N - P = p$, or $P = N - p$, and the general equation becomes $M^2 = 2pN - p^2$. As this equation contains three variable quantities, we may give any values whatever to two of them, and the third will be a function of those two. But if we consider either M or p as the function, the equation will assume the quadratic form, and we cannot attribute all values to the arguments without producing surds. It will therefore be best to consider M and p the arguments, and N the function, or $N = \frac{M^2 + p^2}{2p}$. If we attribute any rational values whatever to M and p, the values of N and $N - p$ will be rational; but our present purpose is to deal only with whole numbers, and we can, therefore, only substitute such integers for M and p as shall render N a whole number. It is evident that any whole number may be used for p, and that M^2 must in all cases be divisible by p. Since $N = \frac{1}{2}\left(\frac{M^2}{p} + p\right)$, $\frac{M^2}{p} + p$ must be an even number; consequently if p is an odd number, $\frac{M^2}{p}$ must be odd, that is, M^2 may be any odd square divisible by p; and if p is an even number, $\frac{M^2}{p}$ must be even, that is, M^2 may be any square divisible by $2p$.

If we make $p = 1$, and regard it as constant, M may be any odd number whatever. When $p = 2$, M^2 must be divisible by 4, and M may be any even number whatever. If $p = 3$, M may be any odd number divisible by 3, and so on. For each value of p we have an infinite series of equations; and since p may have an infinite number of values, we have an infinite number of infinite series. As p is constant for any given series, N and $N - p$ may be readily found by the above formula for any possible value of M. When three equations are thus obtained, the law of the series becomes evident, and the other equations may be obtained without the aid of the formula.

It is worthy of notice that each series, in descending, returns upon itself, — except that M becomes negative, — and each series is thus unlimited in both directions. If the series where $p = 8$ be continued backwards, we have $0^2 = 4^2 - (-4)^2$, $(-4)^2 = 5^2 - (-3)^2$, $(-8)^2 = 8^2 - 0^2$, $(-12)^2 = 13^2 - 5^2$, &c. The formula also shows that M may be made either positive or negative without any change in the other quantities.

As $M^2 + p^2$ must always be positive, it is evident that if we make p negative, N must also be negative (M being either positive or negative, as before), and the series will differ from those produced by the corresponding positive values of p only in having N and $N - p$ take an opposite sign. When $p = 0$, $N = \infty$, and $M^2 = \infty^2 - \infty^2$, unless M is also 0, when $0^2 = N^2 - N^2$. In the two equations last named, M and N represent any finite quantity whatever.

The following table contains the most interesting and important of the first thirty series. Others, up to the thirty-second (where $p = 32$), are merely multiples of those given.

$p = m = 1.$

$1^2 = 1^2 - 0^2$
$3^2 = 5^2 - 4^2$
$5^2 = 13^2 - 12^2$
$7^2 = 25^2 - 24^2$
$9^2 = 41^2 - 40^2$
$11^2 = 61^2 - 60^2$
$13^2 = 85^2 - 84^2$

$p = n = 2.$

$2^2 = 2^2 - 0^2$
$4^2 = 5^2 - 3^2$
$6^2 = 10^2 - 8^2$
$8^2 = 17^2 - 15^2$
$10^2 = 26^2 - 24^2$
$12^2 = 37^2 - 35^2$
$14^2 = 50^2 - 48^2$
$16^2 = 65^2 - 63^2$
$18^2 = 82^2 - 80^2$

$p = n = 8.$

$4^2 = 5^2 - (-3)^2$
$8^2 = 8^2 - 0^2$
$12^2 = 13^2 - 5^2$
$16^2 = 20^2 - 12^2$
$20^2 = 29^2 - 21^2$
$24^2 = 40^2 - 32^2$
$28^2 = 53^2 - 45^2$
$32^2 = 68^2 - 60^2$
$36^2 = 85^2 - 77^2$

$p = m = 9.$

$3^2 = 5^2 - (-4)^2$
$9^2 = 9^2 - 0^2$
$15^2 = 17^2 - 8^2$
$21^2 = 29^2 - 20^2$
$27^2 = 45^2 - 36^2$
$33^2 = 65^2 - 56^2$
$39^2 = 89^2 - 80^2$

$p = n = 18.$

$6^2 = 10^2 - (-8)^2$
$12^2 = 13^2 - (-5)^2$
$18^2 = 18^2 - 0^2$
$24^2 = 25^2 - 7^2$
$30^2 = 34^2 - 16^2$
$36^2 = 45^2 - 27^2$
$42^2 = 58^2 - 40^2$
$48^2 = 73^2 - 55^2$
$54^2 = 90^2 - 72^2$

$p = m = 25.$

$5^2 = 13^2 - (-12)^2$
$15^2 = 17^2 - (-8)^2$
$25^2 = 25^2 - 0^2$
$35^2 = 37^2 - 12^2$
$45^2 = 53^2 - 28^2$
$55^2 = 73^2 - 48^2$
$65^2 = 97^2 - 72^2$

NOTE ON THE "HORIZONTAL THRUST OF EMBANKMENTS."

By Prof. F. W. Bardwell, Antioch College, Yellow Springs, Ohio.

I have taken the earliest opportunity since my return to examine more carefully the question of the horizontal thrust of an embankment of earth rising from the retaining wall in a natural slope, and I am happy to find that my former criticism upon an article by Captain Woodbury was entirely at fault, and that his formula was correct.

I think I understood in the first place the conditions of the problem, but was misled by relying, without sufficient examination, upon a statement in the American edition of Weisbach's Mechanics. The question of the horizontal thrust of an embankment terminating in a horizontal plane is there discussed at length, and the usual formula deduced. But again (Vol. II., page 18, *Remark*) it is stated that this same formula is applicable to the case where the earth is carried above the wall in a natural slope, provided the height of the earth be substituted in the formula for that of the wall. At first this seemed plausible; and, accepting it without further scrutiny, I was afterwards led to criticise too hastily the article of Captain Woodbury.

The formula given by Weisbach is,

$$P = \tfrac{1}{2} h^2 \gamma \operatorname{tang}^2 \left(45° - \frac{\varrho}{2}\right);$$

P being the horizontal thrust, h the height of the wall, γ the density of the earth, and ϱ the angle of friction. To show that this formula is not applicable in the case of a sloping surface, by making h equal to the height of the earth, let us suppose the simple case where the height of the wall is zero, and the earth extends back indefinitely with its natural slope. Now the horizontal thrust is obviously zero, but the above formula would make it indefinitely great.

I have quoted from the American edition of WEISBACH, which contains some additions to the original; but I have not a copy of the German edition at hand to consult with reference to this point.

Let me add, with regard to the paper of Captain WOODBURY, that a careful examination shows the elegance of the method he used, and the simplicity of the formula deduced for its practical application.

NOTE ON THE DEPENDENCE OF EQUATIONS.

By JOHN BORDON, Chicago, Illinois.

A METHOD of deducing equations of a higher order from those of a less degree, by a method of combining the imaginary roots of $x^3 - a^3$, $x^5 - a^5$, &c., has been heretofore shown. The results thereby arrived at can be more easily obtained in some cases by the law of symmetrical functions. For, suppose we have an equation, $x^2 + bx + c = 0$, and we find the coefficients of another equation of the same degree whose roots shall be the cubes of those in the above equation; and suppose $x_{,}^2 + Bx_{,} + C = 0$ to be the equation; then if $-a_{,}, -b_{,}$, are the roots of the first equation, then $B = a_{,}^3 + b_{,}^3$, $C = a_{,}^3 b_{,}^3$. But by symmetrical functions, if we represent by S_1, S_2, S_3, &c., the sum of the first power, sum of the squares, sum of the cubes, &c., of the roots of the first equation, we have

$$S_1 = -b; \quad S_2 = b^2 - 2c; \quad S_3 = -b^3 + 3bc.$$

But $B = -S_3 = +b^3 - 3bc$, and since $c = a_{,} b_{,}$, evidently $C = a_{,}^3 b_{,}^3$, or $c = C^{\frac{1}{3}}$. Hence we have

$$b^3 - 3C^{\frac{1}{3}} b - B = 0.$$

Here we have an equation of the third degree deduced out of two others of the second degree, and by this means we can arrive at

the CARDAN formula. For if the above cubic be the same as $x_{//}^3 + p\,x_{//} + q = 0$, then $x_{//} = b$, $p = -3\,C^{\frac{1}{3}}$, $q = -B$, or $C = -\frac{p^3}{27}$, $B = -q$, and we have

$$x_{/}^2 - q\,x - \frac{p^3}{27} = 0, \quad \text{or} \quad x_{/} = +\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}};$$

but $x_{/} = x^3$, or $x = x_{/}^{\frac{1}{3}}$, and b = sum of the two values of x taken with contrary signs; and as

$$-x = \left(-\frac{q}{2} \pm \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}},$$

then

$$x_{//} = b = \left(-\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}} + \left(-\frac{q}{2} - \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}\right)^{\frac{1}{3}}.$$

So, if we went on obtaining the values of S_4, S_5, S_6, S_7, we would obtain equations of the 4th, 5th, 6th, and 7th degrees, which would be resolvable in the same manner. These would give the forms of equations of those degrees which depend for their solution upon the quadratic. The two modes of solving biquadratics can be arrived at also by symmetrical functions, but the manner of doing so, heretofore pointed out, cannot be simplified. When we arrive at an equation of the 5th degree of the general form $x^5 + p\,x^3 +$ &c., the mode of solving cubics and biquadratics by the use of two equations of the next lower degree becomes more difficult of application. But the equation $x^5 + p\,x^3 +$ &c. can be deduced from two equations of the fourth degree. And it yet remains to be seen whether a solution of the quintic cannot be fairly looked for in that direction. And the general proposition may be stated, that an equation of the form $x^m + p\,x^{m-2} +$ &c. $= 0$ may be derived from two equations of the form

(1) $$x_{/}^{m-1} + A\,x_{/}^{m-2} + \text{&c} = 0,$$

(2) $$x_{//}^{m-1} + a\,x_{//}^{m-2} + \text{&c} = 0;$$

which shall stand to each other in the relation that the roots of Eq. (1) are the mth power of the roots of Eq. (2), i. e. $x_{,,}^{m} = x_{,}$.

And to illustrate this, let us deduce $x^5 + px^3 +$ &c. from two biquadratics. First, we have

$$x_{,}^4 + Ax_{,}^3 + Bx_{,}^2 + Cx_{,} + D = 0, \tag{3}$$

$$x_{,,}^4 + ax_{,,}^3 + bx_{,,}^2 + cx_{,,} + d = 0; \tag{4}$$

in which we suppose $x_{,} = x_{,,}^5$. Now let us arrive at the value of A in terms of a by symmetrical functions: —

$$S_1 = -a; \quad S_2 = a^2 - 2b; \quad S_3 = -a^3 + 3ab - 3c;$$

$$S_4 = a^4 - 4a^2b + 4ac + 2b^2 - 4d;$$

$$S_5 = -a^5 + 5a^3b - 5a^2c - (5b^2 - 5d)a + 5bc.$$

But $$A = -S_5 = a^5 - 5a^3b +$$ &c.

Hence

$$a^5 - 5ba^3 + 5ca^2 + (5b^2 - 5d)a - 5bc - A = 0. \tag{6}$$

Hence, if we have

$$x^5 + px^3 + qx^2 + rx + s = 0, \tag{7}$$

then $a = x$; $p = -5b$; $q = 5c$; $r = 5b^2 - 5d$; $S = -A - 5bc$; that is to say, a root of an equation of the 5th degree, as Eq. (7), will be equal to the sum of the roots of an equation of the fourth degree, as Eq. (4), taken with a contrary sign, when $b = -\frac{p}{5}$; $c = \frac{q}{5}$; $d = \frac{p^2 - 5r}{25}$; $A = \frac{pq}{5} - S$. Now D in Eq. (3) is equal to d^5, hence $D = \left(\frac{p^2 - 5r}{25}\right)^5$; and if B and C were determined in like manner, then x in Eq. (7) would be equal to the sum of the 5th roots of Eq. (3) taken with their signs changed, and the quintic would be resolved into the biquadratic. The law of symmetrical functions renders the general proposition obvious.

NOTE ON NUMERICAL EQUATIONS.

By Prof. A. Vallas, State Seminary and Military Academy, Alexandria, Louisiana.

The Resolution of Numerical Equations, so far at least as real roots are concerned, has been brought to such perfection that scarcely more is expected at present than to render approximations more easy and expeditious.

It is known that the common method of finding the irrational real roots of equations is still that of Newton, although considerably amended and perfected by such men as Budan, Fourier, and Horner.

Geometrically this common method consists in this. The equation being referred to two rectangular axes, and the real roots being the abscissas of the intersections of the horizontal axis by the curve, from an ordinate and trigonometrical tangent of a point near an intersection, to find this intersection by means of the "subtangent."

For in the equation

$$fx = x^m + Ax^{m-1} + Bx^{m-2} + \ldots + Kx + L = 0,$$

put $a + x'$ for x, the results will be,

$$f(a+x') = fa + f'a \cdot x' + \frac{f''a}{1.2}x'^2 + \ldots + \frac{f^{(m)}a}{1.2\ldots m}x'^m = 0.$$

If now a be near one of the real roots, we get, in general, a still nearer value by putting

$$fa + f'a \cdot x' = 0,$$

from which

$$x' = -\frac{fa}{f'a};$$

and the nearer value will be

$$a + x' = a - \frac{fa}{f'a}.$$

That here x' is the subtangent of the point whose abscissa is a, needs not to be proved.

How that, in order to a sure approximation, the root ought first to be included between such limits as contain no singular points, as also how the degree of approximation to be obtained may be determined beforehand, has, with other particulars, been admirably discussed by FOURIER; although it must be regretted that FOURIER despised to connect his disquisitions with those of BUDAN, whose transformed equations (or equations where roots are all less by a) afford very easy means for determining the numerical values of

$$fa,\ f'a,\ f''a,\ f'''a \ldots,$$

which form the substratum of FOURIER's discussion. The two proceedings were at last connected by others, and a method engrafted on this connection, namely, also by HORNER, which may be regarded as the triumph of modern investigation in these matters.

It were needless here to do more than merely mention, that the diminution of roots is, geometrically, equivalent to transposing the axis of the ordinates to the right hand, and that according to this the approximations are obtained by subtangents belonging to points of the curve situated in that axis.

In this note it is proposed to show: 1st, that the approximations aimed at may be more easily obtained by abscissas of parabolic arcs tangent to the curve where it intersects the axis of the ordinates; and, 2d, to give a sure rule (as FOURIER has done for NEWTON's method) for the degree of approximation thus obtained.

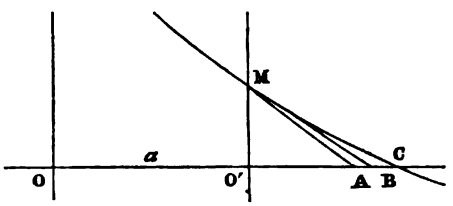
In regard to the first point, let

$$f(a+x') = fa + f'a \cdot x' + \frac{f''a}{1.2} \cdot x'^2 + \ldots = 0$$

be such a transformed equation as excludes all singular points between the point belonging to a and the nearest intersection, and put

$$fa + f'a \cdot x' + \frac{f'a}{1.2} \cdot x'^2 = 0;$$

then will this latter equation manifestly belong to a parabolic curve, having a contact of the second order, at the point belonging to a, with the curve represented by the equation $fx = 0$, or having that contact in the axis of the ordinates, as this axis is supposed to have been removed by as much as a to the right hand.



In the annexed figure, O being the origin of the co-ordinates of the curve represented by $y = fx$, O' will be that of the curve $y' = f(a + x')$. Further, M being the point of the curve in the new axis of the co-ordinates, $O'A$ is the approximation of the nearest intersection by the subtangent, while $O'B$ is that by the abscissa of the parabolic arc in question.

Solving the equation

$$fa + f'a \cdot x' + \frac{f''a}{2} x'^2 = 0,$$

we obtain $$x' = -\frac{f'a}{f''a} \pm \frac{f'a}{f''a} \sqrt{1 - \frac{2fa \cdot f''a}{(f'a)^2}},$$

in which expression the upper sign is to be taken; for, 1st, if x' admits values of different signs, the conditions of the problem exclude the negative one; and, 2d, if both should be positive, those same conditions require us to take the less.

Developing now the expression for x', just found, we get

$$x' = -\frac{fa}{f'a} - \left(\frac{fa}{f'a}\right)^2 \cdot \frac{f''a}{2f'a} - 2\left(\frac{fa}{f'a}\right)^3 \cdot \left(\frac{f''a}{2f'a}\right)^2 \ldots$$

This is the expression for the abscissa of the parabolic arc, or for $O'B$, which abscissa is evidently made up of the subtangent $O'A$ represented by the first, and the portion AB represented by the succeeding terms of the development.

Now, in regard to the degree of approximation, it is evident that in developing in a similar way the value of x'' from

$$fa + f'a \,.\, x'' + \frac{f''(a+\theta x'')}{2} \,.\, x''^2 = 0,$$

which is manifestly the given equation itself; we obtain for $x'' = O'C$,

$$O'C = -\frac{fa}{f'a} - \left(\frac{fa}{f'a}\right)^2 \,.\, \frac{f''(a+\theta x'')}{2f'a} - 2\left(\frac{fa}{f'a}\right)^3 \,.\, \left(\frac{f''(a+\theta x'')}{2f'a}\right)^2 \ldots$$

Or, developing $f''(a + \theta x'')$,

$$O'C = -\frac{fa}{f'a} - \left(\frac{fa}{f'a}\right)^2 \,.\, \frac{f''a}{2f'a} - 2\left(\frac{fa}{f'a}\right)^3 \,.\, \left(\frac{f''a}{2f'a}\right)^2 \ldots$$
$$- \theta x'' \,.\, \left(\frac{fa}{f'a}\right)^2 \,.\, \frac{f'''a}{2f'a} \ldots$$

Leaving out the other terms, we see easily that the difference of the two abscissas $O'C - O'B = BC$ is

$$= -\theta x'' \left(\frac{fa}{f'a}\right)^2 \,.\, \frac{f'''a}{2f'a},$$

$$= -3\theta x'' \,.\, \left(\frac{fa}{f'a}\right)^2 \,.\, \frac{f'''a}{2\,.\,3f'a};$$

and this expression we shall directly discuss.

Suppose the transformed equation to be

$$x'^m + B_{m-1}x'^{m-1} + B_{m-2}x'^{m-2} + \ldots + B_3x'^3 + B_2x'^2 + B_1x' + B_0 = 0;$$

then we shall have

$$B_0 = fa, \quad B_1 = f'a, \quad B_2 = \frac{f''a}{2}, \quad B_3 = \frac{f'''a}{2\,.\,3} \ldots$$

Substituting these values, the difference between the abscissa of the curve represented by the given equation and that of the parabolic arc, or

$$BC \text{ will be } = -3\theta x'' \left(\frac{B_0}{B_1}\right)^2 \,.\, \frac{B_3}{B_1}.$$

If now $$\frac{B_2}{B_1} = \frac{1}{10^m},$$

and we suppose a to be a portion of the root with n correct decimals, that is, x'', and (which is very nearly the same)

$$\frac{B_0}{B_1} = \frac{1}{10^n};$$

we see easily that the difference $O'C - O'B = BC$ will be numerically

$$= < \frac{3}{10^{m+3n}},$$

and we get therefore, in all cases, through the abscissa of the parabolic arc, at least $m + 3n - 1$ correct decimal places.

Let us now return for a moment to the development of this abscissa.

$$O'C = -\frac{fa}{f'a} - \left(\frac{fa}{f'a}\right)^2 \cdot \frac{f''a}{2f'a} - 2\left(\frac{fa}{f'a}\right)^3 \cdot \left(\frac{f''a}{2f'a}\right)^2 \cdots$$

By a similar substitution, the third term becomes

$$-2\left(\frac{B_0}{B_1}\right)^3 \cdot \left(\frac{B_2}{B_1}\right)^2.$$

If we now put $$\frac{B_2}{B_1} = \frac{1}{10^\mu},$$

we shall have $$2\left(\frac{B_0}{B_1}\right)^3 \cdot \left(\frac{B_2}{B_1}\right)^2 = < \frac{2}{10^{2\mu+3n}};$$

hence this term can only affect the $(2\mu + 3n)$th decimal, and may therefore be neglected, when even

$$2\mu \leq m - 1.$$

Having therefore, by diminutions of the roots, sufficiently approached one of the irrational real roots, and having thus obtained a transformed equation of the form

$$x'^m + B_{m-1}x'^{m-1} + \ldots + B_3 x'^3 + B_2 x'^2 + B_1 x' + B_0 = 0,$$

we shall find $x' = -\frac{B_0}{B_1} - \left(\frac{B_0}{B_1}\right)^2 \cdot \frac{B_2}{B_1} - 2\left(\frac{B_0}{B_1}\right)^3 \cdot \left(\frac{B_2}{B_1}\right)^2$

for an approximation correct in $m + 3n - 1$ decimals, if by previous operations n have been found already. And in this it must be remarked, 1st, that m is to be determined by the relation,

$$\frac{B_2}{B_1} < \frac{1}{10^m};$$

and that, 2d, the third term of the value of x' may be omitted whenever $2\mu \leq m$, μ being determined by the relation,

$$\frac{B_2}{B_1} < \frac{1}{10^\mu}.$$

Let us take, for an example, the equation

$$x^3 - 10x^2 - 130x + 858 = 0.$$

It is easy to find that the real roots are contained between 5 and 6, 14 and 15, —10 and —11. In order to find the first of these roots, let us diminish all roots by 5.

5)1	— 10	— 130	+ 850
1	— 5	— 155	+ 75
1	0	— 155	

$$x'^3 + 5x'^2 - 155x' + 75 = 0.$$

It is easy to find the first decimal. According to the common method, there is sometimes a doubt about this. According to our proceeding,

$$-\frac{B_0}{B_1} = \frac{75}{155} = 0.48; \qquad -\left(\frac{B_2}{B_1}\right)^2 \cdot \frac{B_2}{B_1} = \left(\frac{75}{155}\right)^2 \cdot \frac{5}{155} = 0.00;$$

and there is therefore no doubt about this first decimal. Diminishing by 0.4,

0.4)1	5	— 155	75
1	5.4	— 152.84	13.864
1	5.8	— 150.52	

$$x''^3 + 6.2x''^2 - 150.52x'' + 13.864 = 0.$$

Before proceeding further, let us examine how many decimals we shall get by the application of our proceeding.

$$\frac{B_3}{B_1} = \frac{1}{150} < \frac{1}{10^3}, \qquad \frac{B_2}{B_1} = \frac{6}{150} < \frac{1}{10}.$$

We have, therefore, $m = 2$, $\mu = 1$. This will lead us to the fourth decimal inclusively. We shall have

$$-\frac{B_0}{B_1} = 0.09210$$
$$-\left(\frac{B_0}{B_1}\right)^2 \cdot \frac{B_2}{B_1} = \underline{0.00034}$$
$$0.0924$$

The next approximation would give us thirteen correct decimals. If we diminish, however, only by 0.09, we shall obtain at least seven, and this we shall do.

0.09) 1	6.2	— 150.52	13.864
	9	0.5661	— 13.495851
	6.29	— 149.9539	0.368149
	9	0.5742	
	6.38	— 149.3797	
	9		

$$x'''^3 + 6.47\,x'''^2 - 149.3797\,x''' + 0.368149 = 0;$$

from which

$$-\frac{B_0}{B_1} = 0.00246\ 451$$
$$-\left(\frac{B_0}{B_1}\right)^2 \cdot \frac{B_2}{B_1} = \underline{0.00000\ 026}$$
$$0.00246\ 478$$

The root with seven correct decimals is, therefore, 5.4924647.

In applying this method to binomial equations, $x^n + N = 0$, put $x' + a$, instead of x:

$$x'^n + n\,a\,x'^{n-1} + \frac{n(n-1)}{1.2}\,a^2\,x'^{n-2} + \ldots$$
$$+ \frac{n(n-1)}{1.2}\,a^{n-1}\,x' + (a^n + N) = 0.$$

As here $B_0 = a^n + N$, $B_1 = n a^{n-1}$, $B_2 = \frac{n(n-1)}{1.2} a^{n-2} \ldots$,
we get as an approximate value,

$$x' = -\frac{1}{n}\left(a + \frac{N}{a^{n-1}}\right) - \frac{(n-1)}{2a}\left[\frac{1}{n}\left(a + \frac{N}{a^{n-1}}\right)\right]^2 - \frac{(n-1)^2}{2a^2}\left[\frac{1}{n}\left(a + \frac{N}{a^{n-1}}\right)\right]^3 \ldots$$

And we get again at least $m + 3n' - 1$ correct decimals, after n' have been found, m being determined by the relation

$$\frac{(n-1)(n-2)}{1.2} \cdot \frac{1}{a} < \frac{1}{10^m}.$$

If $n = 2$, we obtain the following for the extraction of the square root from the number $-N$:

$$x' = -\tfrac{1}{2}\left(a + \frac{N}{a}\right) - \frac{1}{8a}\left(a + \frac{N}{a}\right)^2 - \frac{1}{16a}\left(a + \frac{N}{a}\right)^3 \ldots$$

In this case, the approximation is unlimited, as $\frac{(n-1)(n-2)}{2.3} = 0$; that is $m = \infty$.

If, for instance, $N = -12$, and $a = 3.46$, we get from the first three terms,

$$-\tfrac{1}{2}\left(a + \frac{N}{a}\right) = +0.00410\ 40462\ 4$$
$$-\frac{1}{8a}\left(a + \frac{N}{a}\right)^2 = -\qquad 24339\ 8$$
$$-\frac{1}{16a}\left(a + \frac{N}{a}\right)^3 = +\qquad 28\ 8$$
$$x' = +0.00410\ 16151\ (4);$$

hence the square root of 12 is 3.46410 16151.

It may be remarked here, that the first term of the development of x', namely, $-\frac{1}{n}\left(a + \frac{N}{a^{n-1}}\right)$, leads to an old rule, according to which, in the extraction of the nth root, nearly twice as many figures are to be found as are contained in a, and which rule is the follow-

ing: "Divide the number by the $(n-1)$st power of the part found (a), subtract the quotient from a, and divide the difference by n." It may also be remarked, that the same first term, as applied to the extraction of the square root, gives this root in nearly twice as many figures by the formula,

$$-\frac{a^2+N}{2a} = -\frac{a^2-N'}{2a} = \frac{N'-a^2}{2a},$$

(as N is supposed to be essentially negative,) which is nothing but a well-known rule, although commonly overlooked.

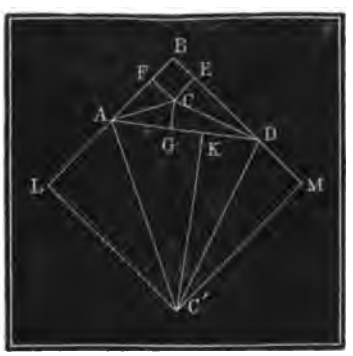
REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. IX., Vol. II.

The first Prize is awarded to Asher B. Evans, Madison Univ., N. Y.
The second prize is awarded to O. B. Wheeler, Michigan University.
The third prize is awarded to J. B. Davison, Oberlin College, Ohio.

Prize Solution of Problem I.

By Miss Harriet S. Hazeltine, Worcester, Mass.

I. The sum of the radii of the inscribed circle and of the circle touching the hypothenuse and extension of the two legs of a right triangle is equal to the sum of the said legs.



Find C and C', the centres of the two circles, by bisecting the angles including the hypothenuse AD, and their supplements. Then CE, CF, CG, perpendiculars to BD, AB, AD, will be radii of the inscribed circle, and $C'K$, $C'L$, $C'M$, perpendiculars to AD and to AB and BD prolonged, will be radii of the circle

whose centre is C'. $C'MBL$ is a square; $\therefore$ radius $C'M = AB + AL = AB + AK = AB + ED$ (Vol. II., No. VI., Problem II.). $CEBF$ is also a square; $\therefore$ radius $CE = EB$, and

$$C'M + CE = AB + ED + EB = AB + BD.$$

Prize Solution of Problem II.

By Miss Harriet S. Hazeltine, Worcester, Mass.

II. The side of a regular octagon, inscribed in a circle whose radius is unity, is $\sqrt{(2-\sqrt{2})}$, and its area is $2\sqrt{2}$. Required the proof.

Denote by AB the side of the inscribed square, and draw the radii CA, CB to its extremities. Draw also the radius CDE perpendicular to and bisecting AB at D, and the subtended arc at E. The chord AE is the side of the regular inscribed octagon. $AB = \sqrt{2}$; $AD = \frac{1}{2}AB = \frac{1}{2}\sqrt{2}$. The angle $CAD = ACD \therefore CD = AD = \frac{1}{2}\sqrt{2}$. $DE = CE - CD = 1 - \frac{1}{2}\sqrt{2}$. $AE = \sqrt{(\overline{AD}^2 + \overline{DE}^2)} = \sqrt{(2-\sqrt{2})}$. Four times the area of the triangle $ABE = 4 \times AD \times DE = 2\sqrt{2} - 2$. Adding this to the area of the inscribed square, which is 2, we shall obtain the area of the inscribed octagon, which is $2\sqrt{2}$.

Prize Solution of Problem III.

By Gustavus Frankenstein, Springfield, Ohio.

III. The perpendicular to the normal of a parabola at its intersection with the principal axis, meets the curve in two points through which tangents are drawn; find the locus of their common point.

Let $x'y'$ be a point of the parabola whose equation is $y^2 = 2px$. We have always the subnormal $= p$ $\therefore$ $p + x'$ is the abscissa of the point of intersection of the normal with the axis of x (or the principal axis of the parabola), and the equation of the line drawn

through this point, perpendicular to the normal, or parallel to the tangent, is, therefore,

$$y = \frac{p}{y'}\,[x - (p + x')].$$

But $by = p(x + a)$ is the equation of the line which joins the points of contact of the two tangents to the parabola which meet in the point a, b. The identity of these equations gives $\frac{p}{y'} = \frac{p}{b}$, or $y' = b$; also, $\frac{-p(p+x')}{y'} = \frac{pa}{b}$, or $p + x' = -a$.

Substituting for x' its value $\frac{y'^2}{2p} = \frac{b^2}{2p}$, there results

$$b^2 = -2pa - 2p^2.$$

The locus is therefore another and equal parabola opening in the opposite direction, with its vertex at the distance p from that of the other.

Either parabola is the locus of the points a, b, in respect to the other parabola.

I find the same reasoning which led me to the solution of this problem, also very readily solves Problem V., No. XI., Vol. I.: — "If through any point, secants be drawn to a curve of the second order, and through the points where each secant intersects the curve, tangents be drawn meeting each other, the locus of the points of meeting is a straight line."

The equation of a line passing through the given point c, d is $y - c = A(x - d)$. The equation of the line which joins the point of contact of the two tangents to a conic section which meets in the point a, b is $2by = (m + 2an)x + am$. The identity of these equations requires that $\frac{m + 2an}{2b} = A$, and $\frac{am}{2b} = c - Ad$,

$$\therefore\ 2bc = (m + 2an)d + am;$$

and thus

$$b = \frac{2nd + m}{2c} \cdot a + \frac{md}{2c}.$$

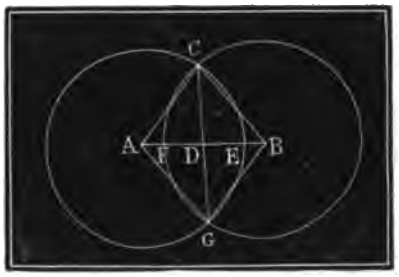
The locus of the points a, b is a straight line easily constructed.

Prize Solution of Problem IV.

By Asher B. Evans, Madison University, Hamilton, N. Y.

IV. Required the radii of three equal circles tangent to each other, and cutting off equal areas from a given circle, so that the sum of these areas may be a maximum.

Let B represent the centre and BC the radius of the given circle, A the centre and AC the radius of one of the three equal tangent circles. Let $BC = 1$, $AC = r$, area of $FCEG = u$; then will



$$AB = \tfrac{2}{3} r\sqrt{3}, \quad AD = \frac{7r^2 - d}{4r\sqrt{3}},$$

$$BD = \frac{r^2 + 3}{4r\sqrt{3}}, \quad CD = \sqrt{\frac{42r^2 - r^4 - 9}{48r^2}},$$

$$CF = \cos^{-1}\left(\frac{r^2 + 3}{4r\sqrt{3}}\right),$$

$$CE = r\cos^{-1}\left(\frac{7r^2 - 3}{4r^2\sqrt{3}}\right).$$

Area, $FCEG = ACEG + BCFG - ACBG$, or

$$(1) \quad u = r^2\cos^{-1}\left(\frac{7r^2 - 3}{4r^2\sqrt{3}}\right) + \cos^{-1}\left(\frac{r^2 + 3}{4r^2\sqrt{3}}\right) - \tfrac{1}{6}\sqrt{(42r^2 - r^4 - 9)}.$$

Differentiating (1) gives

$$(2) \quad \frac{du}{dr} = 2r\cos^{-1}\left(\frac{7r^2 - 3}{4r^2\sqrt{3}}\right) - \frac{1}{3r}\sqrt{(42r^2 - r - 9)}.$$

Putting $\frac{du}{dr} = 0$ gives

$$(3) \quad \cos^{-1}\left(\frac{7r^2 - 3}{4r^2\sqrt{3}}\right) = \frac{2}{\sqrt{3}}\sqrt{1 - \left(\frac{7r^2 - 3}{4r^2\sqrt{3}}\right)^2}.$$

For brevity, put $\frac{7r^2 - 3}{4r^2\sqrt{3}} = x$; then (3) becomes

$$(4) \quad \cos^{-1} x = \frac{2}{\sqrt{3}}\sqrt{(1 - x^2)}.$$

This equation, solved by approximation, gives $x = .60976$, $\therefore r = 1.0385$ nearly.

Equation (2) gives

$$\frac{d^2u}{dr^2} = 2\cos^{-1}\left(\frac{7r^2-3}{4r^2\sqrt{3}}\right) - \frac{12+\frac{3}{r^2}-\frac{1}{3}r^2}{\sqrt{(42r^2-r^4-9)}},$$

which is essentially negative when $r = 1.0385$; hence r is the radius of one of the three equal circles that cut off the greatest areas from the given circle.

No correct solution of Problem V. has been received.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. X., Vol. II.

The first prize is not awarded.

The second prize is awarded to A. G. Barker, Waterville College, Maine.

The third prize is awarded to William Timpson, White Plains, N. Y.

Prize Solution of Problem I.

By A. G. Barker, Waterville College, Maine.

I. Solve the equations

$$x^2y^2(x^4-y^4) = 2340, \qquad xy(xy^3-1)(x^2+y^2) = 1794,$$

by quadratics.

Performing the multiplications indicated, and adding twice the second equation to the first, there results

$$x^6y^2 + 2x^4y^4 + x^2y^6 - 2x^3y - 2xy^3 = 5928;$$

or,
$$x^2y^2(x^2+y^2)^2 - 2xy(x^2+y^2) = 5928.$$

Solving this quadratic, we get

$$xy(x^2+y^2) = 78, \quad \text{or} \quad -76.$$

Combining this result with the given equations, it is easy to find $x = 3$, $y = 2$. These were the only values of x and y found in any of the solutions.

Prize Solution of Problem II.

By William Timpson, White Plains, N. Y.

II. Given the lengths of the three perpendiculars dropped from any point in the plane of an equilateral triangle upon the sides; to find the segments of the sides.

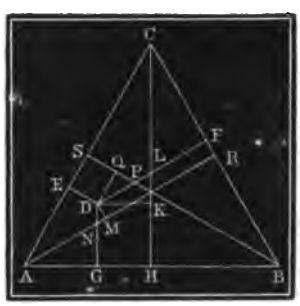
Put $DE = a$, $DF = b$, and $DG = c$. Now,

$$\tfrac{1}{2} AB\,(DE + DF + DG) = \text{area } \triangle ABC = \tfrac{1}{2} AB \,.\, CH;$$

$$\therefore\ CH = DE + DF + DG = a + b + c.$$

Making $AH = x$, $(2x)^2 - x^2 = (a+b+c)^2$, or $2x = \frac{2}{\sqrt{3}}(a+b+c)$, the side of the triangle.

The triangles DKL, CFL, CHB are similar; and since $HB = \frac{1}{2} BC$, $FL = \frac{1}{2} LC$, and $LK = \frac{1}{2} DL$. Let $FL = y$, then $CL = 2y$, $DL = b - y$,



$$LK = \frac{b-y}{2} = a + b - 2y \ \therefore\ y = \frac{2a+b}{3},$$

and $$CL = 2y = \tfrac{2}{3}(2a + b).$$

In the same way, $NG = \frac{2a+c}{3}$, and $AN = \frac{2}{3}(2a+c)$; $PF = \frac{2c+b}{3}$, and $PB = \frac{2}{3}(2c + b)$. Now $FC = \sqrt{(CL^2 - FL^2)} = \frac{2a+b}{\sqrt{3}}$,

$$\therefore\ BF = \frac{2}{\sqrt{3}}(a + b + c) - \frac{2a+b}{\sqrt{3}} = \frac{b+2c}{\sqrt{3}}.$$

Also $AG = \sqrt{(AN^2 - NG^2)} = \frac{2a+c}{\sqrt{3}}$, $\therefore\ GB = \frac{2b+c}{\sqrt{3}}$.

And $$AE^2 + ED^2 = DG^2 + AG^2,$$

$$\therefore\ AE = \frac{a+2c}{\sqrt{3}}, \qquad \text{and} \qquad CE = \frac{a+2b}{\sqrt{3}}.$$

In solving this question, I noticed the following:—

$$AG^2 + GD^2 = AE^2 + ED^2,$$
$$BF^2 + FD^2 = BG^2 + GD^2,$$
$$CE^2 + ED^2 = CF^2 + FD^2,$$
$$\therefore AG^2 + BF^2 + CE^2 = AE^2 + BG^2 + CF^2;$$

or, $$AG^2 - GB^2 + BF^2 - CF^2 + CE^2 - AE^2 = 0;$$

or,

$$(AG+GB)(AG-GB)+(BF+CF)(BF-CF)+(CE+AE)(CE-AE)=0$$

or, $$AB(AG-GB)+BC(BF-CF)+CA(CE-AE)=0.$$

Dividing by $AB = BC = CA$,

$$AG - GB + BF - CF + CE - AE = 0,$$

or, $$AG + BF + CE = GB + CF + AE;$$

that is, the sum of the alternate segments of the sides is equal to one half the perimeter of the triangle.

Prize Solution of Problem III.

III. If e denote the edge of any regular dodecahedron, and $\beta = 36°$, prove that

$$\text{solidity} = \frac{5e^3\cot^2\beta}{2\sqrt{(4\sin^2\beta - 1)}} = \frac{5e^3\tan^2 45}{4\sqrt{(\sin 6°\cos 24°)}}.$$

Also obtain similar formulas for the solidity of the icosahedron.

The following solution is essentially that of Asher B. Evans, Madison University, Hamilton, N. Y.

In any of the regular polyhedrons, let

e = length of an edge.	R = radius of inscribed sphere.
F = number of faces.	r = radius circle inscribed in a face.
n = number of sides to a face.	G = area of a face.
m = number of faces at a solid angle.	V = solidity of the polyhedron.

Let C be the centre of the polyhedron, A the centre of a face, B the vertex of one of the adjacent solid angles, and E the middle point of the adjacent edge.

Let $H =$ area of triangle AEB, and $\alpha =$ angle ACE. Then the angle between the planes ACE and ACB will be $\frac{\pi}{n}$; and that between the planes ACB and BCE will be $\frac{\pi}{m}$; and the intersections of these three planes with the inscribed sphere will form a right spherical triangle, of which one side will be $R\alpha$, and the two angles will be $\frac{\pi}{n}$ and $\frac{\pi}{m}$.

Whence $$\cos \alpha = \frac{\cos \frac{\pi}{m}}{\sin \frac{\pi}{n}}.$$

In the right triangle AEB, the angle at A will be $\frac{\pi}{n}$, and $EB = \frac{1}{2}e$; whence

$$AE = r = \tfrac{1}{2} e \cot \tfrac{\pi}{n}, \qquad H = \tfrac{1}{8} e^2 \cot \tfrac{\pi}{n},$$

$$G = 2nH = \tfrac{1}{4} e^2 n \cot \tfrac{\pi}{n},$$

$$R = r \cot \alpha = \frac{\frac{1}{2} e \cot \frac{\pi}{n}}{\sqrt{\left(\frac{\sin^2 \frac{\pi}{n}}{\cos^2 \frac{\pi}{m}} - 1\right)}},$$

$$V = F \times G \times \tfrac{1}{3} R = \frac{e^3 F n \cot^2 \frac{\pi}{n}}{24 \sqrt{\left(\frac{\sin^2 \frac{\pi}{n}}{\cos^2 \frac{\pi}{m}} - 1\right)}}.$$

For the

Tetrahedron, $F = 4,\ n = 3,\ m = 3,\ V = \frac{e^3\sqrt{2}}{12}$.

Cube, $F = 6,\ n = 4,\ m = 3,\ V = e^3$.

Octahedron, $F = 8,\ n = 3,\ m = 4,\ V = \frac{e^3\sqrt{2}}{3}$.

Dodecahedron, $F = 12,\ n = 5,\ m = 3,\ V = \frac{5 e^3 \cot^2 36°}{2\sqrt{(4 \sin^2 36° - 1)}}$

$$= 5e^3 \sqrt{\left(\frac{47 + 21\sqrt{5}}{40}\right)} = \frac{5 e^3 \tan 45°}{4\sqrt{(\sin 6° \cos 24°)}}.$$

Icosahedron, $F = 20,\ n = 3, m = 5,\ V = \frac{5 e^3 \cot^2 60°}{2\sqrt{\left(\frac{\sin^2 60°}{\cos^2 36°} - 1\right)}} = 5e^3\left(\frac{3+\sqrt{5}}{12}\right)$.

Mr. EVANS's solution of Problem IV. is vitiated by a slight oversight in the last case.

No correct solution of Problem V. has been received, probably owing to a misapprehension of the problem. The bisection referred to in the problem is general, and independent of any law of variation in density. Hence a plane cannot be the bisecting surface.

SIMON NEWCOMB,
W. P. G. BARTLETT,
TRUMAN HENRY SAFFORD.

DISCUSSION OF THE EQUATIONS WHICH DETERMINE THE POSITION OF A COMET OR OTHER PLANETARY BODY FROM THREE OBSERVATIONS.

By G. W. HILL, Nyack Turnpike, N. Y.

THESE equations may be found in the *Mécanique Céleste*, Vol. I., p. 207, or in PONTECOULANT, *Théorie Analytique du Système du Monde*, Vol. II., p. 21, where the analysis conducting to them is given. The equations are

$$\rho = \frac{R^4}{m}\left(\frac{1}{r^3} - \frac{1}{R^3}\right), \qquad r^2 = R^2 - 2R\rho\cos c + \rho^2;$$

in which ρ is the comet's distance from the earth, r its radius vector, R the earth's radius vector, c the angular distance of the comet from the sun, and m a function of known quantities. The unknown quantities are ρ and r. Now, to find the values of ρ and r, it has always been advised to determine them from the equations in the above form by the method of double position. But this way is needlessly complex in calculation, and we shall give the equations a much simpler form, to which the tentative process can be easily applied. Assume $r = \frac{R\sin c}{\sin\theta}$, and $\rho = \frac{R\sin(\theta - c)}{\sin\theta}$; these values substi-

tuted in the last equation render it identical; and from the first we obtain

$$\frac{\sin^3\theta}{\sin^3 c} = 1 + m\,\frac{\sin(\theta - c)}{\sin\theta};$$

or, $$\sin^4\theta = \sin^3 c\,[(1 + m\cos c)\sin\theta - m\sin c\cos\theta].$$

Let $\sin^3 c\,(1 + m\cos c) = A\cos B$, and $m\sin^4 c = A\sin B$; whence

$$\tan\beta = \frac{m\sin c}{1 + m\cos c}, \quad \text{and} \quad A = \frac{m\sin^4 c}{\sin\beta} = \frac{\sin^4 c}{\sin(c - \beta)}.$$

Then the equation becomes

$$\sin^4\theta = A\sin(\theta - \beta).*$$

This is probably the most elegant form to which the above equations can be reduced. Taking the logarithms of both sides, we have

$$4\log\sin\theta - \log\sin(\theta - \beta) = \log A.$$

From this a near approximate value of θ can be found by inspection of the table of logarithmic sines, without any calculation but that which can be performed mentally. The exact value may then be obtained by a single application of the method of double position.

We proceed to notice some properties of the roots of the equation to which we have reduced the two first given.

The equation may be put under this form:

$$\sin^8\theta - 2A\cos\beta\sin^5\theta + A^2\sin^2\theta - A^2\sin^2\beta = 0.$$

From the powers of $\sin\theta$, which are wanting, we perceive that the equation has at least four imaginary roots; and the sign of the last term being negative, there are at least two real roots, one positive, the other negative. By reference to the equations which determine ρ and r, it will be seen that they are satisfied by the values $\rho = 0$,

* See Gauss's *Theoria Motus*, and especially the Appendix to Commander Davis's translation. — Ed.

and $r = R$; the corresponding value of θ is c, which, put in the equation $\sin^4\theta = A\sin(\theta - \beta)$, renders it identical. Since, by the nature of the problem only positive values of ρ and r are admissible, and consequently θ being contained between the limits c and π, we may reject as useless the four imaginary roots, and the negative root and the root $\theta = c$. There remain two roots which are necessarily real, because the problem must have at least one solution, and we are led to the important conclusion that it cannot have more than two.

Let us here deduce a relation which exists between the known quantities when these two roots are real. Differentiating the equation $4\log\sin\theta - \log\sin(\theta - \beta) = \log A$ with respect to θ, we get for the equation containing the values of θ, when

$$4\log\sin\theta - \log\sin(\theta - \beta)$$

is a maximum or minimum,

$$4\cot\theta - \cot(\theta - \beta) = 0, \quad \text{or} \quad 4\tan(\theta - \beta) = \tan\theta;$$

from which we derive

$$\tan\theta = \frac{3 \pm \sqrt{(9 - 16\tan^2\beta)}}{2\tan\beta}.$$

If $\tan^2\beta > \frac{9}{16}$, this value of $\tan\theta$ is imaginary, and the expression $4\log\sin\theta - \log\sin(\theta - \beta)$ is not susceptible of a maximum or minimum value, and the equation in $\sin\theta$ has only two real roots, which are among those rejected. Hence we conclude, that when the data are taken from observation, the quantity $\frac{m\sin c}{1 + m\cos c}$ will always be contained between the limits $\pm\frac{3}{4}$. If we substitute 0 for θ in the equation $\sin^4\theta - A\sin(\theta - \beta) = 0$, the result is $A\sin\beta$, and for $\theta = \pi$ the result is $-A\sin\beta$; showing the existence of an odd number of roots between the limits $\theta = 0$ and $\theta = \pi$, which odd number is three, since c and the root which the

problem demands are within these limits. If we make $\theta - c + dc$, there results the quantity $(4 \sin^3 c \cos c - A \cos(c - \beta))\, dc$; or,

since $\quad A = \frac{\sin^4 c}{\sin(c-\beta)}, \qquad \tan\beta = \frac{m \sin c}{1 + m \cos c},$

the quantity $(3 \cos c - m) \sin^3 c\, dc$. And $-A \sin\beta$, the result on putting $\theta = \pi$, is equal to $-m \sin^4 c$. Therefore, $\sin c$ being always positive, θ has two real values, or only one (between the limits c and π), and consequently the problem two or one answer, according as m and $m - 3 \cos c$ have the same or different signs.

It is evident that $A \cos\beta$ must be positive, in order that the equation in $\sin\theta$ may have three real positive roots; so the quantity $1 + m \cos c$ is always positive, and $\tan\beta$ has the same sign as m. If β be taken between the limits $\pm \frac{\pi}{2}$, A is always positive. Since the equation in $\sin\theta$ must have no root greater than one, units substituted for $\sin\theta$ in the first derived function of its equation must render it positive; that is, the expression $4 - 5A \cos\beta + A^2$ is positive, which gives $A < 2$, and $A \cos\beta < \frac{4}{5}$. According as m is positive or negative, the equation for finding θ presents itself under two shapes, $\sin^4\theta = A \sin(\theta - \beta)$, and $\sin^4\theta = A \sin(\theta + \beta)$, in which A is always positve and less than 2, and β never exceeds 36° 53′.

From the expression for ρ in terms of r, it is clear that r is less or greater than R, according as m is positive or negative. Therefore, in the first case, θ is contained between the limits c and $\pi - c$; and in the second case, if c is in the first quadrant, between $\pi - c$ and $\pi - \beta$; but if c be in the second quadrant, between c and $\pi - \beta$. These remarks may be of use to shorten the tentative process of finding θ.

With regard to θ, it is clear it is the angle subtended at the comet by its radius vector and the line joining it and the earth prolonged beyond the comet.

Mathematical Monthly Notices.

Elements of Chemical Physics. By JOSIAH P. COOKE, JR., Erving Professor of Chemistry and Mineralogy, in Harvard University. Boston: Little, Brown, & Co. 1860. 8vo. pp. 739.

This is the first of an expected series of three volumes, to form "an extended work on the Philosophy of Chemistry." It is, however, in itself a complete treatise of the principles involved in the processes of weighing and measuring.

The introduction gives a view of the nature of physical and chemical laws as seen by the light of a metaphysical theory of causation, rather doubtful in itself, but very popular among modern theologians.

In the second chapter, under the head of "General Properties of Matter," we have an account of the English and French measures of length, and the elementary principles of mechanics, including some formulæ for the motion of the pendulum. From some of the reasoning and conclusions respecting *vis-viva* in this chapter, we are disposed to dissent, as being inaccurate, and the term "centre of gravity" seems to be sometimes used in a very unusual sense.

Chapter III., entitled "The Three States of Matter," gives a highly interesting, extended, and luminous account of the characteristic properties of solids, liquids, and gases, with many valuable tables. Upwards of sixty pages of this chapter are devoted to crystallography, profusely illustrated by diagrams on a black ground; and eighty pages to the mutual molecular actions between matter in its three forms, taken two and two: e. g. between solids and solids, solids and liquids, etc.

The fourth chapter is devoted to the *chemical* theory of heat; and seems to be much more full and satisfactory than the chapters on that subject in any work previously published in this country. It includes accounts of the results of REGNAULT and DULONG on the specific heat and expansion of gases, of PERSON'S theory of the absolute zero of heat, and of JOULE'S experiments on the mechanical theory of heat, so interesting to the physical philosophers; all the results being deduced mathematically, where necessary, and expressed quantitatively, where it is possible. It contains a vast and patiently collected mass of information.

As a general text-book for the student, it labors under the disadvantage of not completely filling the course of either physics or chemistry, as usually taught in colleges; but we think the third and fourth chapters might with some advantage, and a great deal of pleasure to the student, be substituted for so much of the course of chemistry as treats of the properties of the -yles, -ines, -ones, etc. The style of these chapters is so free, the illustrations so excellent and numerous, and the typography so good, that no one possessed of the slightest taste for physical science could fail to be interested by them. To the special student in chemistry the work must be both interesting and valuable, from its exhaustive character.

Elements of Analytical Geometry and of the Differential and Integral Calculus. By CHARLES DAVIES, LL.D., Professor of Higher Mathematics, Columbia College. 2 vols. in one. New York: A. S. Barnes and Burr. 1860. pp. 204, 194.

That there is a growing interest in the subjects treated in these volumes there can be no doubt. Their study has been gradually working its way into seminaries, academies, and even into many high-schools. In these institutions a large proportion of students end their course of study, and for such the work before us is mainly intended. In view of such a use, the manner of presenting and the arrangement of the propositions, and especially what and how much to retain out of such abundance, become very serious questions. In these respects,

as well as in the clearness and simplicity with which the work is written, the author has been remarkably successful. The discussion of the general equation of the second degree is brief, but very clear and satisfactory. The arrangement is as follows: — Introduction. Book I. Points and Lines in a Plane. Books II. – V. Of the Circle, Ellipse, Parabola, Hyperbola. Book VI. Space, Point, and Line, Surfaces of the First and Second Orders. It is enough to say of this part of the volume, that it is a judicious abridgment of the Author's Analytical Geometry.

But Part Second, to which we wish to call more particular attention, shows even more marked evidence of thought and care in arrangement and detail. With the arrangement we are especially pleased, and believe it will receive the approbation of all practical teachers. In Section I. we find Definitions and First Principles; II. Differentials of Algebraic Functions; III. Integration and Applications; IV. Successive Differentials, &c.; V. Maxima and Minima; VI. Differentials of Transcendental Functions; VII. Transcendental Curves; and, lastly, Integration of Rational Fractions, Integration by Parts, and a Discussion of the Cases in which Binomials can be integrated. It will be seen that the processes of differentiation and integration are combined, instead of the usual division. The integration of the various classes of functions immediately follows their differentiation, and then the related application of both processes. The advantages of this combination of the two converse processes are obvious; and especially when the student has but little time to devote to the subject.

We sincerely regret that we cannot give the same hearty assent to the author's exposition of the first principles of the subject. For brevity, and to be sure of presenting his theory completely, we quote, retaining his italics. "Where no law of change has been fixed, such a law may be imposed as will make the variable to change uniformly, and to pass through all values between any two limits. There exists a constant difference between two consecutive values, which forms no appreciable part of the quantity itself." — *Introduction.*

"If the change is *uniform*, and such as to embrace *every value* between given limits, such change cannot be expressed by a number whose unit is any part of 1. For, however small the decimal denoting the change, it might still be diminished by prefixing ciphers; hence, contrary to the supposition, there would be values which h did not pass through. Thus we cannot express the change by numbers, and a new language becomes necessary." § II.

"Before N can coincide with M, there must be a *last* value of $x' - x$; we denote it by dx; and x, $x + dx$, are called *consecutive*. *Consecutive values admit of no intermediate value under the same law of change*; hence their difference cannot be expressed by a number, but is, *numerically* $= 0$. It is named a *differential.*" § 13.

"dx is smaller than any number; its numerical value is 0. $(dx)^2$ is 0, compared with dx; i. e. infinitely less than dx." § 20.

We infer (1) that *before* $x' - x$ *vanishes*, it has a "last value" which really exists, but which is too small to be further diminished by prefixing ciphers; and that all so-called continuous change is actually composed of an infinity of such steps; (2) that while dx is such a *minimum possibile*, yet dy is smaller if the derivative $\frac{dy}{dx}$ be between -1 and $+1$; and that $(dx)^2$ is infinitely less than dx. This is the meaning, unless the proviso "under the same law of change" modifies it in some way which we do not clearly understand, and which may well perplex the young student for whom the book is written. Now (1) is manifestly untrue; it is also incompatible with (2). But were both (1) and (2) true, we think two objections to the theory would remain. First, it involves the mystery of continuity, and also the inconceivable idea of *absolute* infinitesimals or infinites of different orders; while the calculus requires us to discuss neither. Second, it treats differentials as magnitudes *essentially fixed*, while the spirit of the Calculus is to regard them as *essentially arbitrary*. We prefer to define them as any changes

taken so small that the character of the result could not be changed by taking them still smaller; and derivatives as the *limits* toward which $\frac{dy}{dx}$, &c. finally *approach*, as dx approaches zero.

The author demonstrates the theorems of MACLAURIN and TAYLOR separately, and gives no hint that the two are identical. But by making $a = 0$, TAYLOR'S development of $f(a+x)$ in powers of x, becomes MACLAURIN'S development of $f(x)$; while putting $\phi(x) = f(a+x)$, MACLAURIN'S development of $\phi(x)$ becomes TAYLOR'S development of $f(a+x)$.

We have thus, as fully as space will permit, pointed out what we consider as defects of, in other respects, an admirable work; one having too many positive merits not to be improved by just adverse criticism.

Elements of Geometry, with Practical Applications to Mensuration. By BENJAMIN GREENLEAF, A. M., Author of "The National Arithmetic," "Treatise on Algebra," &c. Fourth Electrotype Edition. Boston: Published by Robert S. Davis & Co. 1859.

This work is the last of the series yet published, beginning with the "National Arithmetic," which was first issued in 1835. We understand that a work on Plane and Spherical Trigonometry is soon to be added. The "National Arithmetic," by frequent revisions, has been kept fully up with the times. The last revision, in 1857, has left it an excellent work, especially for fitting students for business life, and its clearly expressed rules will always make it a good book for reference, as well as text-book.

The Geometry is based on that of LEGENDRE, and for the sake of comparison we append the table of Contents. Book I. First Principles. II. Ratio and Proportion. III. The Circle, and the Measure of Angles. IV. Proportions, Areas, and Similarity of Figures. V. Problems Relating to the preceding Books. VI. Regular Polygons, Area of Circles. VII. Planes, Diedral and Polyhedral Angles. VIII. Polyhedrons. IX. The Sphere and its Properties. X. The Three Round Bodies. XI. Applications of Geometry to the Mensuration of Plane Figures. XII. Applications to Solids. XIII. Miscellaneous Geometrical Exercises. XIV. Application of Algebra to Geometry. This will show to what extent the excellent work of LEGENDRE has been reproduced in the one before us. In the different books we notice many changes in the order of the propositions, and a few in the methods of demonstration; and many of them appear to us as the result of experience in the school-room, and as better adapting the work to the wants of beginners. But in our opinion its crowning excellence is the clearness and conciseness with which the definitions and demonstrations are expressed. The greatest care has evidently been taken with the style; and after a careful search we have not been able to find one expression which we felt sure could be improved. Book XIII., comprising undemonstrated propositions, and problems for construction and numerical solution, is another valuable feature, and the only fault is that they are too few. For instance, only ten of the undemonstrated propositions relate to Book I., when there ought to have been at least thirty-five, one to each of the demonstrated propositions. Besides, these propositions and problems would be more likely to be used if they had been inserted in their proper places throughout the book, just what should have been done with Books XI. – XIX. A book on Arithmetic or Algebra would not be tolerated with so few exercises, and all put at the end. There are advantages in not bringing in the definitions before they are wanted. If the definitions of parallels had been based on the idea of direction, as that of an angle is, the change from LEGENDRE would, in our opinion, have simplified several of the earlier propositions much more than that adopted by the author has done. The changes we have suggested are matters of opinion, and by no means imply that the work is not one which we can heartily recommend to the attention of teachers. Indeed, we believe that teachers will generally introduce all the exercises and applications in their proper places, and thus remove our only objection to the arrangement.

Supplemens au traité de mécanique céleste

Depuis l'impression du traité de mécanique céleste, j'ai publié sur les divers objets de cet ouvrage, un grand nombre de recherches insérées dans les mémoires de l'institut et de l'académie des sciences et dans la connoissance des tems. je vais ici réunir ces recherches dans l'ordre qui me semble le plus convenable, en leur donnant toute la généralité dont elles m'ont paru susceptibles. je vais commencer par ce qui concerne la figure des corps célestes, et les oscillations des fluides qui les recouvrent.

Autographe de Laplace.

Cette note, écrite de la main du célèbre géomètre français a été donnée à Mr. Watson par Madame la Marquise de Laplace sa veuve, qui en a certifié en ma présence la parfaite authenticité.

Ch. Delaunay

membre de l'institut de France.

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PRIZE PROBLEMS FOR STUDENTS.

I. Prove that the surface of a sphere can be completely covered with the surfaces either of 4, or of 8, or of 20 equilateral spherical triangles.

II. Which is greater, $\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+$ &c., *ad inf.*, or $\frac{1}{3}+\frac{1}{9}+\frac{1}{27}+$ &c., *ad inf.*?

III. $$\frac{(x+a+b)^5+(x+c+d)^5}{(x+a+c)^5+(x+b+d)^5}=\frac{m}{n}.$$

Find x by quadratics.

IV. A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article, thereby gaining at the rate of 11 per cent more on his outlay than he would gain if the balance were *true*. If however the scale-pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the article. Find the legitimate gain per cent on the article.

V. Show that in the parabola the half sum of the radii vectores drawn to the extremities of any arc whatever, is equal to the radius vector drawn to the summit of the diameter passing through the centre of the chord and parallel to the axis, plus the part of this diameter intercepted between the arc and the chord.— Communicated by Asaph Hall, Esq.

Solutions of these problems must be received by January 1, 1860.

REMARKS ON THE PROBLEM OF THE LIGHTS.*

By Prof. W. H. Parker, Middlebury College, Vermont.

I am very much obliged to you for showing, in your editorial remarks on my "Note" in the February number, page 160, Vol. I., that the error which I conceived to be one of interpretation, really consists in a fallacy, to which the misinterpretation is only incidental.

Having re-examined the subject from a new point of view, suggested by your remarks, and with particular reference to the question, how can we determine from our equations whether the conditions of a problem are impossible or not? I desire, with your permission, to add the following remarks.

The general equation of our problem is

$$\text{(A)} \qquad \frac{a}{x^2} = \frac{b}{(c-x)^2},$$

whose roots are $\frac{c\sqrt{a}}{\sqrt{a} \pm \sqrt{b}}$. If in the roots we suppose $c = 0$, both values of $x = 0$. In order to determine whether we have here the true roots of our equation, under the particular supposition, let us go back to equation (A), and change it to the form which will express the condition $c = 0$. We have now

$$\text{(B)} \qquad \frac{a}{x^2} = \frac{b}{x^2},$$

the same form which we should have obtained by stating the particular problem which this enunciates. Our roots 0 do not satisfy this equation. A moment's consideration is sufficient to show why they should not. So long as a and b are unequal, the expression

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* These Remarks reply to some incidental points in our article, the correctness of the main question being admitted. It is proper to state that this article was received shortly after the publication to which it refers, and has been delayed for want of space. — Ed.

$\frac{a}{x^2} = \frac{b}{x^2}$ is not a true equation; for we have unequals a and b divided by the same quantity, x^2, and no value of x (not even $x = 0$) will make the members equal. This shows conclusively the impossibility of the conditions. When, therefore, our problem requires us to find the point or points of equal illumination of two unequal lights, which are placed together, it requires what the algebraic enunciation of the conditions shows to be impossible.

In like manner, if we take the form of the problem in the editorial remarks, whose equation is

$$\text{(C)} \qquad \frac{a}{(x-\alpha)^2} = \frac{b}{(\beta-x)^2},$$

and whose roots are $\frac{\beta\sqrt{a} \pm \alpha\sqrt{b}}{\sqrt{a} \pm \sqrt{b}}$; and in the roots suppose $\beta = \alpha$, we obtain $x = \alpha$. To determine whether α is a true root, let us go back to equation (C), from which it was derived. Making $\beta = \alpha$, we get

$$\text{(D)} \qquad \frac{a}{(x-\alpha)^2} = \frac{b}{(\alpha-x)^2},$$

a form of equation which α will not satisfy. Hence α is not one of the roots. Indeed the expression $\frac{a}{(x-\alpha)^2} = \frac{b}{(\alpha-x)^2}$ has no roots, for the reason that it is not an equation. For here, too, we have unequals a and b divided by equals; and here, just as in the former case, we find the conditions to be impossible, because we see that the expressions for the intensities cannot be equal.

With the knowledge that the expressions $\frac{a}{x^2} = \frac{b}{x^2}$ and $\frac{a}{(x-\alpha)^2} = \frac{b}{(\alpha-x)^2}$ are not equations at all, it is absurd for us to attempt their reduction, or to ask for their roots. From false premises, surely, we cannot expect reliable conclusions.

I deem it important to remark here, that, since 0 is not a true root of equation (B), nor α of equation (D), we have in neither of them the symbol of infinity. How, then, is the impossibility of the conditions made known? Simply, as it seems to me, by the untruth-

fulness of the equations; or, in other words, by the impossibility of forming the conditions into real equations.

The idea that an impossibility in the conditions will always manifest itself somewhere by the symbol of infinity, I cannot think to be well founded. In the cases already considered, we have an impossibility without the symbol of infinity. On the other hand, we may have the symbol, in a true equation, without any impossibility in the conditions. This is made apparent by considering the case in which $a = b$ and $c = 0$ in equation (A), producing the form $\frac{a}{x^2} = \frac{a}{x^2}$. This is an identical equation, and of course true for any value of x, including $x = 0$. When $x = 0$, we have in each member the symbol of infinity, though we have no impossibility in the conditions. For when the lights are together, and of equal intensity, the point where they are placed is truly a point of equal illumination.

If in equation (C) we make $a = b$ and $\beta = \alpha$, it comes $\frac{a}{(x-\alpha)^2} = \frac{a}{(\alpha-x)^2}$, which is also an identical equation, and, as before, a case of real indetermination. The equation is true for any value of x, including $x = \alpha$. But when $x = \alpha$, each member is a symbol of infinity; and yet here we have no impossibility, because when the equal lights are placed together at the distance α from the origin, the point they occupy is equally illuminated by them.

It will be observed that I have not been speaking of the symbol of infinity in the *roots* of an equation.

The error of BOURDON seems to me to have originated in his failing to observe that the supposition $c = 0$, in the equation of the problem, destroys the equality of the members, and makes his premises false. Reasoning from them as if true, which is his first error, he falls into another at the point indicated in the editorial remarks, page 179; not precisely because, in any form of the problem, the particular *hypothesis* introduces a factor *zero* (since α is not a true root of (D), nor 0 of (B)), but because, in reducing the expression

(B) as if it were an equation, he obtains a form ($a x^2 = b x^2$, or $(a - b) x^2 = 0$) whose members would be equal, if $x = 0$, and hence infers that x is equal to 0 in form (B); thus surreptitiously admitting the factor *zero*, rather than legitimately introducing it by the particular hypothesis.

In a true equation, I do not discover that the presence of *zero* as a factor leads to any fallacy, so long as the factor remains. The removal of a factor *zero* from a real equation may introduce a fallacy, but it is one which the algebraist is apt to discover if he has inadvertently fallen into it.

One word in reference to a seeming antagonism between my remarks on the reliability of the algebraic processes as a method of reasoning, and the editorial remarks in regard to the "*implicit fallacy* to which algebraic transformations are often liable." The difference I suppose to be rather apparent than real. I have been accustomed to regard the fallacies which sometimes occur in our use of the algebraic methods as resulting from errors in ourselves. When our first equation is true, and our transformations are legitimately made, i. e. in strict accordance with those necessary truths which underlie all changes of form in equations, I have supposed we might always rely implicitly upon our conclusions. In this confidence I can hardly think myself singular.

If it be true, that there are no fallacies inherent in the algebraic processes, is it not better to speak of the fallacies to which *we* are often liable in algebraic transformations, than to use a form of expression which seems inconsistent with the absolute integrity of those processes as a method of reasoning?

The above remarks are submitted, in the hope that, if they fail to illustrate the topics discussed, they will at least furnish the occasion for remarks from yourself or others, which shall bring the truth to light.

NOTE ON THE PROBLEM OF THE LIGHTS.

By Hugh Godfray, St. John's College, Cambridge, England.

May I be allowed to make a few remarks on the difficulty in the interpretation of the Problem of the Two Lights, noticed by Professor Parker in Vol. I., page 160, of the Monthly?

Let us consider the problem somewhat differently, and suppose the two lights together at the outset.

$$D' \qquad Q' \qquad O \qquad P \quad Q \qquad D$$

Let two lights, A, B, of intensities a, b, at unit of distance, be placed together at O in the line $D'D$, and let a point, P, which receives light from A, move from D through O towards D'; we can find the corresponding positions of a point, Q, which moves in such a way as to receive from B the same illumination as P does from A. (We suppose b to have a finite ratio to a.)

The equation connecting the distances will be

$$\frac{a}{OP^2} = \frac{b}{OQ^2};$$

whence

$$OQ = \pm OP\sqrt{\frac{b}{a}},$$

which shows that there are two positions, Q, Q', at equal distances to the right and left of O corresponding to each position of P.

Let us confine our attention to the $+$ sign, which gives that position of Q which is on the same side of O as P.

$$OQ = OP\sqrt{\frac{b}{a}}.$$ *

* If we suppose that OQ and OP are both absolute *zeros*, that is, that the two lights occupy absolutely the same point, then the equation $OQ = OP\sqrt{\frac{b}{a}}$ is not true, because $\frac{OQ}{OP} = 1$, and not the constant ratio of $\sqrt{\frac{b}{a}}$. A light of a fixed intensity at a unit's distance must illuminate

We see that OQ is in a constant ratio to OP. When OP diminishes, OQ diminishes; when OP becomes negative, or P passes to the left of O, OQ also becomes negative, and Q passes to the left of O, the two points P and Q passing through O simultaneously, a result which agrees with that of the particular case in the usual way of considering the problem; but perhaps the above method of arriving at it may tend to explain away the difficulty experienced in the other.

It is true that when the lights are of unequal intensity there is no point on either side of O equally illuminated; for if OP has any finite value, OQ has a different value; but the distance between these two points, which is

$$OP\,\frac{\sqrt{b}-\sqrt{a}}{\sqrt{a}},$$

diminishes with OP, and vanishes when QP vanishes, whatever be the relative intensities of A and B.

The value *zero*, therefore, in the usual problem is neither superfluous nor erroneous. The apparent contradiction involved in the statement, that if we admit this value we must also admit that the point O is at the same time equally and unequally illuminated, will disappear if we consider that when the point P is indefinitely near to O, the point Q, which is equally illuminated, is also indefinitely near to O; and points at which the illumination from B is 2, 3, 4, &c. times that at P will at the same time, as can be shown in a similar manner, be indefinitely near to O, all these points coinciding with O at the same instant that P does.

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the point it occupies by a fixed amount, if it illuminates this point at all; and unless we can arbitrarily change the intensity, without producing a corresponding change in the amount by which this point is illuminated, the ordinary solution of the problem for the special case when the two lights are together must be untrue.

We have already shown (page 178, Vol. I.), that, under the hypothesis of the lights occupying the same point, the problem fails altogether, it being impossible to satisfy the corresponding equation with *zero*, or any other value of x. — Ed.

The problem as solved by BOURDON and others would therefore be incomplete without the value *zero*, which I consider a perfectly legitimate solution of the particular case.

The very fact of putting a problem into algebraic language invests it with all the generality the symbols we introduce admit of, and the results of our investigations are nearly always more general and complete than we anticipated in the particular question we proposed to ourselves; and if difficulties of interpretation or superfluous solutions arise, they will be found to have been introduced in the process of reduction, or to be due to the impossibility of expressing in the equation all the restrictions and limitations of the problem, which is not the case in the present instance.

AN EASY MODE OF APPROXIMATING TO THE TIME OF VIBRATION IN A CIRCULAR ARC.

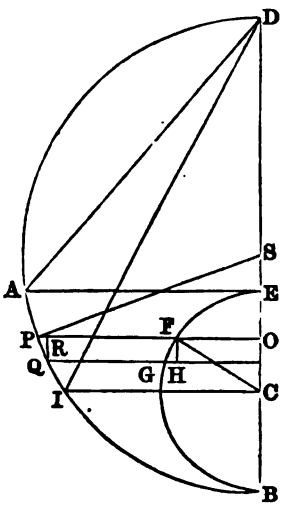
By PROF. LEWIS R. GIBBES, College of Charleston, South Carolina.

LET a particle descend a circular arc AB, whose centre is S, urged by a uniformly accelerating force, acting always parallel to DB, the vertical diameter of the circle; it is required to find T, the time of descent, the arc being supposed not more than a quadrant.

Through A, the point from which the descent begins, draw the horizontal line AE; on EB the versed sine of the arc of descent AB as a diameter, describe the semi-circle EFB, whose centre is C; let PQ be an element of the arc AB, and draw the other lines as shown in the figure, whose construction will be easily understood.

Suppose the particle descending from A to have reached the point P, and about to describe the element PQ in the next element of time, with the acquired velocity continued uniformly; put f to express the intensity of the accelerating force, or the additional velocity

acquired in each second of time; also r for the radius SP or SB, or length of the simple pendulum vibrating in the given arc; $\sqrt{(2f \times EO)}$ will express the velocity acquired in descending AP, with which velocity the element PQ is described.



From the elementary triangle PQR, and the similar one SPO, we have

$$PQ \div QR = SP \div PO;$$

from the elementary triangle FGH, and the similar one FCO, we have

$$FH \div FO = FG \div FC;$$

by properties of the circle, we have

$$FO \div \sqrt{(EO \times OB)} = PO \div \sqrt{(DO \times OB)};$$

also write the identity $\sqrt{2f} = \sqrt{2f}$;

and the equality $SP = \sqrt{(\frac{1}{2} DB \times SB)}$.

Hence, observing that $FH = QR$, we obtain

$$\frac{PQ}{\sqrt{(2f \times EO)}} = \frac{FG}{FC} \times \sqrt{\frac{\frac{1}{2} SB}{2f}} \times \sqrt{\frac{DB}{DO}}.$$

This may be written

$$\frac{\text{elementary arc } PQ}{\sqrt{(2f \times EO)}} = \frac{\text{elementary arc } FG}{2FC} \times \sqrt{\frac{r}{f}} \times \sqrt{\frac{DB}{DO}}.$$

Now, since $\frac{\text{elementary arc } PQ}{\sqrt{(2f \times EO)}} = \frac{\text{space}}{\text{velocity}}$ = the element of time in which is described the elementary space PQ, if $\sqrt{\frac{DB}{DO}}$ were a constant, we could write, the sum of the elements of time employed in descending the successive elements of space of which PQ is one =

$$\frac{\text{sum of the elementary circular arcs of which } FG \text{ is one}}{\text{diameter of the circle}} \times \sqrt{\frac{r}{f}} \times \sqrt{\frac{DB}{DO}},$$

and since the sum of the elements of time constitutes T, the whole time of descent down AB, and the sum of the elementary circular arcs, of which FG is one, constitutes the semi-circumference EFB, and $\frac{\text{semi-circumference}}{\text{diameter}}$ is equal to $\frac{\pi}{2}$, we might write

$$T = \tfrac{\pi}{2}\sqrt{\tfrac{r}{f}} \times \sqrt{\tfrac{DB}{DO}}.$$

But $\sqrt{\frac{DB}{DO}}$ has a different value for every such elementary arc as FG, and we require, not the sum of the elementary arcs, but the sum of the products of each elementary arc by the corresponding value of $\sqrt{\frac{DB}{DO}}$; this summation constitutes the difficulty of the problem, and may be effected by the use of arcs of the ellipse and hyperbola. In its place we will substitute an approximation by using a *mean value* of $\sqrt{\frac{DB}{DO}}$.

The greatest value of $\sqrt{\frac{DB}{DO}}$ is $\sqrt{\frac{DB}{DE}}$, and the least is $\sqrt{\frac{DB}{DB}} = 1$, the arithmetical mean of these values is $\frac{1}{2}\left(1 + \sqrt{\frac{DB}{DE}}\right)$, which, substituted in the equation for T, gives

$$T = \tfrac{\pi}{2}\sqrt{\tfrac{r}{f}} \times \tfrac{1}{2}\left(1 + \sqrt{\tfrac{DB}{DE}}\right).$$

By the properties of the circle, $\sqrt{\frac{DB}{DE}} = \frac{DA}{DE}$, and this ratio is the secant of the angle ADB, or the secant of half the arc AB; if this arc AB, the arc of excursion of the pendulum, be A, then

$$T = \tfrac{\pi}{2}\sqrt{\tfrac{r}{f}} \times \frac{1 + \sec \frac{1}{2}A}{2},$$

as an approximate value for the time. The subjoined table gives, for every 10° of arc A in the first column, the corresponding value of $\frac{1 + \sec \frac{1}{2}A}{2}$, in the second column.

Another mode of approximation presents itself. The greatest

value of DO is DB, and the least is DE, and we may take as its mean value DC, which corresponds to the centre of the circle described on EB as a diameter, and substitution gives

$$T = \frac{\pi}{2}\sqrt{\frac{r}{f}} \times \sqrt{\frac{DB}{DO}};$$

A.	$\frac{1+\sec \frac{1}{2}A}{2}$	a.	$\sec \frac{1}{2}a$.	Mean of 2d & 4th Columns.
°		° ′		
10	1.001910	7.32	1.001904	1.001907
20	1.007713	14.06	1.007618	1.007666
30	1.017638	21.05	1.017168	1.017813
40	1.032089	28.00	1.030614	1.031351
50	1.050689	34.47	1.047954	1.049822
60	1.077350	41.25	1.069011	1.073181
70	1.110387	47.51	1.093929	1.102159
80	1.152704	54.04	1.122659	1.137681
90	1.207107	60.00	1.154701	1.180904

as $\frac{DB}{DO} = \frac{DI}{DO}$ = secant of IDC, or the secant of half the arc IB, if this arc be expressed by a, we shall have $T = \frac{\pi}{2}\sqrt{\frac{r}{f}} \times \sec \frac{1}{2}a$. The arc a may be found from A, the given arc of excursion by either of the following equations: —

$$\textit{versed sine}\ a = \tfrac{1}{2}\ \textit{versed sine}\ A; \qquad \cos a = \cos^2 \tfrac{1}{2}A;$$
$$\sin \tfrac{1}{2}a = \sin \tfrac{1}{2}A \sin 45°.$$

If EB, the height of the arc of descent AB, be expressed by h, then $\textit{versed sine}\ a = \frac{h}{2r}$. The table gives, for every 10° of arc A in the first column, the corresponding value of arc a in the third column, and of sec $\frac{1}{2}a$ in the fourth.

As there is nothing in the investigation to indicate which method will give the closest approximation to the truth, we may take the arithmetical mean of the two as a final result, and we obtain

$$T = \frac{\pi}{2}\sqrt{\frac{r}{f}} \times \frac{1+\sec \frac{1}{2}A+2\sec \frac{1}{2}a}{4},$$

as a final approximate expression for the time of descent in a circular arc of A degrees, A being not more than 90°. The fifth column of the table gives the mean of the second and fourth columns, which corresponds to the last factor in the preceding expression. As neither of the two approximations differs from the mean by half a unit in the fifth decimal place when $A = 10°$, either may be used for cal-

culating T when A is less than 10°; when A is 5°, neither will differ from the mean in the sixth decimal place. The first will be the most convenient expression when A is given; the second when h is given.

The double of either of the three expressions for T will of course be the time of one complete vibration of the pendulum, whose length is r and angle of excursion A. When A is very small, the last factor in each of the above expressions will differ very little from unity, and we may then put $T = \frac{\pi}{2}\sqrt{\frac{r}{f}}$, the double of which is the time of vibration in the arc of a cycloid whose axis is vertical and equal to $\frac{r}{2}$. The geometrical construction used above can easily be applied to the cycloid, to deduce this result.

It may, perhaps, be assumed with safety, that, for small values of A, our final result does not differ from the truth more than the two first approximations do from each other; but, for large values of A, its error can only be determined by comparing it with the true value of T, furnished by the analytic treatment of the problem. As this value is expressed by $\frac{\pi}{2}\sqrt{\frac{r}{f}}$ multiplied by a certain series, we must expand the last factor in each of our expressions above into a series, and then we need only compare the series together, as the quantity $\frac{\pi}{2}\sqrt{\frac{r}{f}}$ is common to all the expressions. Since DB is $2r$, $DE = 2r - h$, $DC = 2r - \frac{1}{2}h$, in the first mode of approximation,

$$\tfrac{1}{2}\left(1 + \sqrt{\frac{DB}{DE}}\right) = \tfrac{1}{2}\left(1 + \sqrt{\frac{2r}{2r-h}}\right) = \tfrac{1}{2}\left(1 + \left(1 - \frac{h}{2r}\right)^{-1}\right),$$

and in the second mode,

$$\frac{DB}{DC} = \sqrt{\frac{2r}{2r - \frac{1}{2}h}} = \left(1 - \tfrac{1}{2}\cdot\frac{h}{2r}\right)^{-1};$$

developing these expressions by the binomial theorem, we shall have

the first and second series given below (the third line contains the arithmetical mean of the two), which series is rewritten on the fourth line, slightly altered in form for readier comparison with the true series, which is written on the fifth line. In the general term, n indicates the order, beginning with the second.

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}\left(\tfrac{h}{2r}\right)+\tfrac{1}{2}\cdot\tfrac{1.3}{2.4}\left(\tfrac{h}{2r}\right)^2+\tfrac{1}{2}\cdot\tfrac{1.3.5}{2.4.6}\left(\tfrac{h}{2r}\right)^3+\tfrac{1}{2}\cdot\tfrac{1.3.5.7}{2.4.6.8}\left(\tfrac{h}{2r}\right)^4+\&c.$$

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}\left(\tfrac{h}{2r}\right)+\tfrac{1}{2^2}\cdot\tfrac{1.3}{2.4}\left(\tfrac{h}{2r}\right)^2+\tfrac{1}{2^3}\cdot\tfrac{1.3.5}{2.4.6}\left(\tfrac{h}{2r}\right)^3+\tfrac{1}{2^4}\cdot\tfrac{1.3.5.7}{2.4.6.8}\left(\tfrac{h}{2r}\right)^4+\&c.$$

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}\left(\tfrac{h}{2r}\right)+\tfrac{1}{2}\cdot\tfrac{2+1}{2^2}\cdot\tfrac{1.3}{2.4}\left(\tfrac{h}{2r}\right)^2+\tfrac{1}{2}\cdot\tfrac{2^2+1}{2^3}\cdot\tfrac{1.3.5}{2.4.6}\left(\tfrac{h}{2r}\right)^3+\tfrac{1}{2}\cdot\tfrac{2^3+1}{2^4}\cdot\tfrac{1.3.5.7}{2.4.6.8}\left(\tfrac{h}{2r}\right)^4+\&c.$$

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\left(\tfrac{h}{2r}\right)+\tfrac{1.3}{2.4}\cdot\tfrac{1.3}{2.4}\left(\tfrac{h}{2r}\right)^2+\tfrac{1.3.5}{2.4.6}\cdot\tfrac{1.3.5}{2.4.6}\left(\tfrac{h}{2r}\right)^3+\tfrac{1.3.5.7\frac{1}{2}}{2.4.6.8}\cdot\tfrac{1.3.5.7}{2.4.6.8}\left(\tfrac{h}{2r}\right)^4+\&c.$$

$$1+\tfrac{1}{2}\cdot\tfrac{1}{2}\cdot\left(\tfrac{h}{2r}\right)+\tfrac{1.3}{2.4}\cdot\tfrac{1.3}{2.4}\cdot\left(\tfrac{h}{2r}\right)^2+\tfrac{1.3.5}{2.4.6}\cdot\tfrac{1.3.5}{2.4.6}\cdot\left(\tfrac{h}{2r}\right)^3+\tfrac{1.3.5.7}{2.4.6.8}\cdot\tfrac{1.3.5.7}{2.4.6.8}\left(\tfrac{h}{2r}\right)^4+\&c.$$

On comparing the last two lines, it will be seen that the series furnished by our third or final approximation does not diverge from that furnished by analysis, until the term containing the *fourth power* of the variable $\frac{h}{r}$ the versed sine of A, and errs in excess; yet for $A = 60°$, it gives a result *true to the sixth decimal place*, and, even for $A = 90°$, gives an excess of only half a unit in the third decimal place; that is, of only one five-hundredth of the whole. It is therefore abundantly sufficient for the determination of the variation in the rate of a clock depending on the variation in the extent of the arc of vibration.

The idea of using an approximation to avoid the difficulties of the problem is not new, nor is the geometric construction, adopted in the demonstration, presented as novel, although we are unable to say who first used it; but we have never seen it handled in such a manner as to secure the advantages that belong to the geometric method when the subject is within its scope. Sometimes the calculus is applied to the geometric construction, and then the indepen-

dence and comprehensiveness of the calculus is sacrificed in the combination. We have endeavored so to exhibit the subject, as to secure the simplicity and directness which are the characteristics of the geometric method, and our three modes of approximation, which seem very naturally to present themselves, are, as far as we are aware, new,— never having fallen in our way,— and, with the aid of the usual trigonometrical tables, furnish a far readier means of computing the time in a circular arc than does the series given by the calculus.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. XI., Vol. II.

The first Prize is awarded to Asher B. Evans, Madison Univ., N. Y.

The second prize is awarded to John A. Winebrener, Princeton College, N. J.

The third prize is awarded to E. O. Gibson, South New Berlin, N. Y.

Prize Solution of Problem I.

Prove that an equation of the second degree cannot have more than two roots.

The general equation of the second degree is

$$Ax^2 + Bx + C = 0.$$

If this equation can have more than two roots, say three, denote them by a, b, c; then

$$Aa^2 + Ba + C = 0, \quad Ab^2 + Bb + C = 0, \quad Ac^2 + Bc + C = 0.$$

By subtraction, and dividing by the factors $a - b$, $a - c$, $b - c$, which by hypothesis are not zero, we get

$$A(a+b) + B = 0, \quad A(a+c) + B = 0, \quad A(b+c) + B = 0.$$

By subtracting again,

$$A(b-c)=0, \quad A(a-c)=0, \quad A(a-b)=0.$$

But these equations are impossible, since A is not zero. Hence, the given quadratic cannot have three different roots.

Prize Solution of Problem II.

By E. O. Gibson, South New Berlin, New York.

The sum of n numbers in arithmetical progression is a, and the sum of their squares is b. Required, the first term and the common difference.

Let x denote the first term, and y the common difference; then

(1) $x+(x+y)+(x+2y)+\ldots.+(x+(n-1)y)=a.$

(2) $x^2+(x+y)^2+(x+2y)^2+\ldots.+(x+(n-1)y)^2=b;$

or, by reduction,

(3) $$nx+\tfrac{1}{2}n(n-1)y=a,$$

(4) $$nx^2+n(n-1)xy+\tfrac{1}{6}n(n-1)(2n-1)y^2=b.$$

From n times (4) subtract the square of (3); we get

$$\frac{n^2(n^2-1)}{12}y^2=bn-a^2. \quad \therefore y=\pm\frac{2}{n}\sqrt{\left(\frac{3(bn-a^2)}{n^2-1}\right)}.$$

By substituting this value of y in (3), we find

$$x=\frac{a}{n}\pm\frac{1}{n}\sqrt{\left(\frac{3(n-1)(bn-a^2)}{n+1}\right)}.$$

Prize Solution of Problem III.

Given, three lines, a, b, c, drawn from any point within a regular polygon of n sides to any three of its consecutive corners, A, B, C. Required, the side of the polygon, and also a geometrical solution, granting that a polygon of n sides can be constructed when one side is given.

Upon the line AB, taken equal to (b), construct the polygon of (n) sides, $ABCDEF$ (in the figure $n=6$), and join two alternate corners, as A and C, by the diagonal AC. From A as a centre,

with a radius equal to (a), describe an arc, and taking C as a centre, with a radius equal to (c) describe another arc, cutting the first at P, taking care that P and B line on opposite sides of the line AC. Join PB, and upon it describe the polygon $BPIKLM$, which is the one required. For, by construction, $AP = a$ and $AB = b$. Also, the angle $CBA = PBM$, since they are angles of regular polygons of the same number of sides. Subtract PBA, and $CBP = ABM$; also, by construction, $CB = AB$ and $PB = BM$ $\therefore$ $AM = CP = c$.

Schol. — If P lie on the *same* side with B, of the line AC (as at P'), then will BP' be the side of another polygon of n sides, which will fulfil all the conditions of the problem, except that the point A, from which the three lines are drawn, lies *outside* of it.

This solution, by Prof. C. A. Young, Western Reserve College, Hudson, Ohio, is neater than any received, and therefore we give it.

Prize Solution of Problem IV.

By all the Competitors.

Deduce the formulæ of Arithmetical and Geometrical Progressions by the method of differences.

If d_1, d_2, d_3, &c. denote the successive orders of difference, a the first term and s the sum of n terms, then the usual formula is

$$(1) \qquad S = na + \frac{n(n-1)}{1.2} d_1 + \frac{n(n-1)(n-2)}{1.2.3} d_2 + \text{&c.}$$

In an arithmetical progression the first differences are constant, and therefore d_2, d_3, &c. equal zero; hence

$$S = na + \frac{n(n-1)}{1.2} d = \frac{n}{2}(a + a + (n-1)d) = \frac{n(a+b)}{2}.$$

In a geometrical progression,

$$S = a + ar + ar^2 + \ldots.$$

$$d_1 = a(r-1), \quad d_2 = a(r-1)^2, \quad d_3 = a(r-1)^3, \text{ \&c.}$$

Therefore (1) becomes

$$S = na + \frac{n(n-1)}{1.2} a(r-1) + \frac{n(n-1)(n-2)}{1.2.3} a(r-1)^2 + \text{\&c.}$$

$$= \frac{a}{r-1}\left\{ n(r-1) + \frac{n(n-1)}{1.2}(r-1)^2 + \frac{n(n-1)(n-2)}{1.2.3}(r-1)^3 + \text{\&c.} \right\}$$

$$= \frac{a}{r-1}\left\{ (r-1+1)^n - 1 \right\} = \frac{a(r^n-1)}{r-1},$$

since the series in { } lacks 1 of being the nth power of $1 + (r-1)$.

Prize Solution of Problem V.

By John A. Winebrener, Princeton College, New Jersey.

If three points on the surface of a sphere (radius = 1) be joined by three arcs of small circles whose angular radii are ϱ, ϱ', ϱ'', the area of the included triangle will be

$$\theta + \theta' + \theta'' + \pi \pm \varphi \cot \varrho \pm \varphi' \cot \varrho' \pm \varphi'' \cot \varrho'';$$

in which φ, φ', φ'' are the lengths of these arcs, and θ, θ', θ'' the angles at which they intersect.

Let ABC be the given triangle. Take any point P within it, and join PA, PB, PC. Let O be the pole of BC, or φ, and join OB, OC. Since $\sin \varrho$ is the planar radius of φ, we have by geometry

$$(1) \quad \angle BOC : \varphi = 1 : \sin \varrho. \quad \therefore \angle BOC = \frac{\varphi}{\sin \varrho}$$

The surface of a zone, of which OBC is a part, equals $2\pi(1 - \cos \varrho)$. Therefore the surface of

$$(2) \qquad OBC = \frac{\varphi}{\sin \varrho}(1 - \cos \varrho).$$

Denote the angles of the triangle PBC by $\alpha, \alpha', \alpha''$ · then the surface of

$$(3) \qquad PBOCP = \alpha + \alpha' + \alpha'' - \pi - \frac{\varphi}{\sin \varrho},$$

since $\angle PBC = \alpha' - 90°$, &c.

Adding (2) and (3) the surface of

$$(4) \qquad PBC = \alpha + \alpha' + \alpha'' - \pi - \varphi \cot \varrho.$$

The triangles APB and APC will give similar results, and by adding these, and noting that the angles $A + B + C = \theta + \theta' + \theta''$, and the angles at $P = 2\pi$, we have finally the surface of

$$(5) \quad ABC = \theta + \theta' + \theta'' - \pi \pm \varphi \cot \varrho \pm \varphi' \cot \varrho' \pm \varphi'' \cot \varrho''.$$

Note by Asher B. Evans. — If $\theta_1, \theta_2, \ldots \theta_n$; $\varphi_1, \varphi_2, \ldots \varphi_n$; $\varrho_1, \varrho_2, \ldots \varrho^n$ represent similar quantities for a polygon formed on the surface of a sphere (radius = 1), by joining n points in order, by arcs of small circles, then the area of this polygon will be

$$(6) \quad \theta_1 + \theta_2 + \ldots \theta_n - (n-2)\pi \pm \varphi_1 \cot \varrho_1 \pm \varphi_2 \cot \varrho_2 \pm \ldots \varphi_n \cot \varrho_n.$$

If $n = 3$, (5) and (6) are identical; if $n = 2$, (6) becomes

$$\theta_1 + \theta_2 \pm \varphi_1 \cot \varrho_1 \pm \varphi_2 \cot \varrho_2 = 2\theta_1 \pm \varphi_1 \cot \varrho_1 \pm \varphi_2 \cot \varrho_2,$$

which is the area of a lune formed by the intersection of two small circles, whose radii are ϱ_1 and ϱ_2, angle of intersection θ_1, φ_1 and φ_2 denoting the lengths of the arcs.

Simon Newcomb,

W. P. G. Bartlett,

Truman Henry Safford.

NOTES AND QUERIES.

Note on the Greatest Common Measure.

I. The difference between two multiples of a number must be a multiple of that number.

II. Any times a multiple of a number must be a multiple of that number.

Admit the above, and the following solutions are easily uuderstood.

1. Find the greatest common measure of 6 and 28; 72 and 486; 252 and 462; 1573 and 3553; 1181, 2741, and 3413.

2. Mathematical Monthly, Vol. I. p. 5, Ex. 2. "Required the greatest common divisor of 1633, 3763, 4757, and 4189."

Solution of No. I.

$$6)28(5 \quad 72)486(7 \quad 252)462(2 \quad 1573)3553(2$$
$$30 \qquad 504 \qquad 504 \qquad 3146$$
$$2 \text{ Ans.} \qquad 18 \text{ Ans.} \qquad 42 \text{ Ans.} \qquad 407)1573(4$$
$$1628$$
$$55 = 5.11.$$

Ans. 11.

$$1181)2741 \quad 3413$$
$$2362 \quad 3543$$
$$379 \quad 130 = 10.13.$$

Neither 2, 5, nor 13 measures 379.

∴ Ans. 1.

Solution of No. II.

$$1633)3763 \quad 4757 \quad 4189$$
$$3266 \quad 4899 \quad 4899 \quad 1633$$
$$497 = 7.71 \quad 142 = 2.71 \quad 710 = 10.71 \quad 71 = 23.$$

Ans. 71.

The advantage, in many cases, of taking a quotient figure too large by unity, which may be done in any stage of the process, is readily seen. The products could be written over the dividend, if thought necessary. — E. Hunt, Sub-Master English High School, Boston.

Notes from Todhunter's Algebra.

I. *Determine the conditions which make the binomial* $a \pm \sqrt{b}$ *a perfect cube.*

If we assume

(1) $\sqrt[3]{(a+\sqrt{b})} = x + \sqrt{y}$; then (2) $\sqrt[3]{(a-\sqrt{b})} = x - \sqrt{y}$.

For $a + \sqrt{b} = (x+\sqrt{y})^3 = x^3 + 3x^2\sqrt{y} + 3xy + y\sqrt{y}$; and, since the rational parts, as well as irrational parts, of this equation are equal,

$\therefore a = x^3 + 3xy$, $\sqrt{b} = 3x^2\sqrt{y} + y\sqrt{y}$; $\sqrt[3]{(a-\sqrt{b})} = x - \sqrt{y}$.

The product of (1) and (2) is

$$(3) \qquad \sqrt[3]{(a^2-b)} = x^3 - y = c,$$

if $a^2 - b$ is a perfect cube, and $= c$.

But $a = x^3 + 3xy = x^3 + 3x(x^2 - c)$ from (3); and therefore $4x^3 - 3cx = a$, from which x may be found; and then y is known from the equation $y = x^3 - c$.

II. *Find the sum of the squares of the first* n *natural numbers.*

We know that $1 + 2 + 3 + \ldots + n = \frac{n(n+1)}{2}$.

Let $S = 1^2 + 2^2 + 3^2 + \ldots + n^2$; then since

$$\begin{aligned} n^3 - (n-1)^3 &= 3n^2 - 3n + 1, \\ (n-1)^3 - (n-2)^3 &= 3(n-1)^2 - 3(n-1) + 1, \\ (n-2)^3 - (n-3) &= 3(n-2)^2 - 3(n-2) + 1, \end{aligned}$$

and so on to

$$\begin{aligned} 3^3 - 2^3 &= 3 \cdot 3^2 - 3 \cdot 3 + 1, \\ 2^3 - 1^3 &= 3 \cdot 2^2 - 3 \cdot 2 + 1, \\ 1^3 - 0^3 &= 3 \cdot 1^2 - 3 \cdot 1 + 1, \end{aligned}$$

we get, by adding vertically,

$$n^3 = 3(1^2 + 2^2 + 3^2 + \ldots + n^2) - 3(1 + 2 + 3 + \ldots + n) + n$$
$$= 3s - \frac{3n(n+1)}{2} + n, \quad \therefore s = \frac{n(n+1)(2n+1)}{2 \cdot 2}.$$

III. *Find the sum of the cubes of the first* n *natural numbers.* We have

$$n^4 - (n-1)^4 = 4n^3 - 6n^2 + 4n - 1,$$
$$(n-1)^4 - (n-2)^4 = 4(n-1)^3 - 6(n-1)^2 + 4(n-1) - 1,$$
$$(n-2)^4 - (n-3)^4 = 4(n-2)^3 - 6(n-2)^2 + 4(n-2) - 1,$$

and so on to

$$3^4 - 2^4 = 4 \cdot 3^3 - 6 \cdot 3^2 + 4 \cdot 3 - 1,$$
$$2^4 - 1^4 = 4 \cdot 2^3 - 6 \cdot 2^2 + 4 \cdot 2 - 1,$$
$$1^4 - 0^4 = 4 \cdot 1^3 - 6 \cdot 1^2 + 4 \cdot 1 - 1,$$

and, by adding vertically, we get

$$n^4 = 4(1^3 + 2^3 + 3^3 + \ldots n^3) - 6(1^2 + 2^2 + 3^2 + \ldots n^2) + 4(1 + 2 + 3 + \ldots n) - n.$$
$$= 4s - \frac{6n(n+1)(2n+1)}{2 \cdot 3} + \frac{4n(n+1}{2} - n.$$
$$\therefore 4s = n^4 + 2n^3 + n^2, \quad \text{or} \quad s = \left(\frac{n(n+1)}{2}\right)^2.$$

It follows, therefore, that

$$(1 + 2 + 3 + \ldots + n)^2 = 1^3 + 2^3 + 3^3 + \ldots + n^3.$$

The sums of the fourth, fifth, &c. powers of the first n natural numbers may, in succession, be found by the same process.

IV. *To find the value of a recurring decimal.* Let P denote the p figures which do not recur, Q the q figures which do recur, and s the value of the recurring decimal; then

$$s = .PQQQ\ldots, \quad 10^p s = P.QQQ\ldots, \quad 10^{p+q} s = PQ.QQ\ldots$$

and by subtraction $(10^{p+q} - 10^p)\, s = PQ - P \ldots s = \frac{PQ - P}{10^{p+q} - 10^p}$.
Now, $10^{p+q} - 10^p = 10^p(10^q - 1)$; and $10^q - 1$, when expressed by figures, in the usual way, will consist of q *nines.*

Hence we find the usual rule for finding the value of a recurring decimal. *Subtract the integral number consisting of the non-recurring figures*

from the integral number consisting of the non-recurring and recurring figures, and divide by a number consisting of as many nines as there are recurring figures followed by as many ciphers as there are non-recurring figures.

V. *Demonstration of the Binomial Theorem for any exponent*, supposing the development for positive integer exponents to be known.

Suppose m and n to be positive integers; then

$$(1)\ (1+x)^m = f(m) = 1 + mx + \frac{m(m-1)}{1.2}x^2 + \frac{m(m-1)(m-2)}{1.2.3}x^3 + \&c.$$

$$(2)\ (1+x)^n = f(n) = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \&c.$$

using the notation $f(m)$ and $f(n)$ for brevity. The product of (1) and (2) gives

$$(1+x)^m (1+x)^n = (1+x)^{m+n} = f(m) f(n).$$

But, since $m+n$ is a whole number, the development of $(1+x)^{m+n}$ must be of the same form as (1) or (2).

$$(3) \qquad \therefore (1+x)^{m+n} = f(m+n) = f(m) \times f(n).$$

For n write p, and then change m to $m+n$; then

$$f(m+n+p) = f(m+n) f(p) = f(m) \times f(n) \times f(p).$$

Continuing this process, we shall obtain

$$(4)\quad f(m+n+p+q..) = f(m) \times f(n) \times f(p) \times f(q) \times \ldots$$

Now, let $m = n = p = q = .. = \frac{s}{r}$, a fraction, s and r being positive integers, and suppose the number of fractions to be r; then (4) becomes

$$f(s) = \left(f\left(\frac{s}{r}\right)\right)^r. \qquad \therefore (f(s))^{\frac{1}{r}} = f\left(\frac{s}{r}\right).$$

But since s is a positive integer, $f(s) = (1+x)^s$, and therefore

$$\left(f(s)\right)^{\frac{1}{r}} = (1+x)^{\frac{s}{r}} = f\left(\frac{s}{r}\right) = 1 + \frac{s}{r}x + \frac{\frac{s}{r}\left(\frac{s}{r}-1\right)}{1.2}x^2 + \&c.$$

when the exponent is any positive fraction. Again, in (3) put $-n$ for m; then

$$f(-n) \times f(n) = f(-n+n) = f(0) = 1,$$

as is evident from (1) or (2).

Wherefore $\frac{1}{f(n)} = f(-n)$. But for any positive value of n we have $f(n) = (1+x)^n$.

$$\therefore \frac{1}{f(n)} = \frac{1}{(1+x)^n} = (1+x)^{-n} = f(-n)$$
$$= 1 + (-n)x + \frac{(-n)(-n-1)}{1.2}x^2 + \&c.,$$

which shows the form of the development to be the same for negative exponents. This demonstration for fractional and negative exponents is due to EULER.

EXPOSITION OF THE PROCESS OF MATHEMATICAL DEVELOPMENT.

By JOHN PATERSON, A. M., Albany, N. Y.

[Continued from Page 294, Vol. II.]

6. DURING the time $1'_t$, while the generating force φ^{n-2} of the third rank has increased duplicately from 0 to the ultimate value $3\varphi^{n-2}$, its generated force of the fourth (descending) rank, φ^{n-3}, has increased in triplicate ratio; so that if $3\varphi^{n-2}$ has generated $1\varphi^{n-3}$ in the time $1'_t$, it will generate $3\varphi^{n-3}$ in the time $1''_t$. Applying this relation to the semi-units $\frac{1}{2}'.1'_t$ and $\frac{1}{2}''.1'_t$, instead of the units $1'_t$ and $1''_t$ of time, and recollecting that since φ^{n-3} increases in triplicate ratio during the time of its genesis, its effect in an equal time with its ultimate value constant must necessarily be m'' times its effect, while increasing triplicately from zero, it is plain that the following equations are authorized.

$$\tfrac{1}{2}'.1_t' + \tfrac{1}{2}''.1_t' = 1_t':$$

$$\tfrac{1}{8}\varphi_1^{n-3} + \tfrac{2}{8}\varphi_1^{n-3} + \tfrac{2}{8}\varphi_2^{n-3} + \tfrac{1}{8}\varphi_2^{n-3} = 1\,\varphi^{n-3};$$

$$\frac{m''}{8}\varphi_1^{n-3} + \frac{3\,m''}{8}\varphi_1^{n-3} + \frac{3\,m''}{8}\varphi_2^{n-3} = \frac{m''}{8}\varphi_2^{n-3} = m''\varphi^{n-3};$$

the first and last terms having the relation of constant to variable, as before.

The force $\frac{m''}{8}\varphi_1^{n-3}$ would generate $\frac{m''}{8}\varphi^{n-4}$ in a unit of time, and therefore does generate $\frac{m''}{16}\varphi^{n-4}$ in the time $\frac{1}{2}''.1_t'$, the m''th part of which is $\frac{1}{16}\varphi^{n-4}$ the amount it has generated in the time $\frac{1}{2}'.1_t'$ of its own genesis, and is also the amount generated by $\frac{m''}{8}\varphi_2^{n-3}$ in the time $\frac{1}{2}'.1_t'$ of its genesis.

The force $\frac{3\,m''}{8}\varphi_1^{n-3}$ would generate $\frac{3\,m''}{8}\varphi^{n-4}$ in a unit of time, or $\frac{3\,m''}{16}\varphi^{n-4}$ in a semi-unit of time; but as it is of the third rank from its constant generator $\frac{2}{2}\varphi_1^{n-1}$, it increases in duplicate ratio, and therefore its successive effect must be divided by 3 to get its synchronous measure $\frac{2}{3}.\frac{m''}{16}\varphi^{n-4}$ for the time $\frac{1}{2}''.1_t'$.

The force $\frac{3\,m''}{8}\varphi_2^{n-3}$ would generate $\frac{3\,m''}{8}\varphi^{n-4}$ in a unit of time, or $\frac{3\,m''}{16}\varphi^{n-4}$ in a semi-unit of time; but as it is of the second rank from its constant generator $\frac{2}{2}\varphi_1^{n-2}$, it increases in simple ratio, and therefore its successive effect must be divided by 2 to get $\frac{2}{2}\frac{m''}{16}\varphi^{n-4}$, its synchronous effect in the time $\frac{1}{2}''.1_t'$. The collected terms fetch the equation $\left(\frac{1}{16} + \frac{2}{8}\frac{m''}{16} + \frac{2}{2}\frac{m''}{16} + \frac{1}{1}\frac{m''}{16} + \frac{1}{16}\right)\varphi^{n-4} = 1\,\varphi^{n-4}$, which gives $m'' = 4$,

$$\tfrac{1}{2}'.1_t' + \tfrac{1}{2}''.1_t' = 1_t':$$

and converts the last preceding equation into

$$\tfrac{4}{8}\varphi_1^{n-6} + 3.\tfrac{4}{8}\varphi_1^{n-3} + 3.\tfrac{4}{8}\varphi_2^{n-3} + \tfrac{4}{8}\varphi_2^{n-6} = 4\,\varphi^{n-3}.$$

Thus the successive and synchronous measures of a force of the fourth rank are mutually convertible by the factor 4.

NOTE. — A quantity may increase from zero by a regular law to any assigned magnitude in a given time; and it is seen by the preceding deductions that the law of increase determines the divisor for reducing successive to synchronous measure, independently of the absolute magnitude of the former. In the last preceding case, the successive semi-unit measure of the effect of the force $3 \cdot \frac{4}{8} \varphi_2^{n-3}$ is $3 \cdot \frac{4}{16} \varphi^{n-4}$; but as this effect increases in duplicate ratio, its synchronous measure calls for the divisor 2; while in the same place the successive semi-unit effect of $3 \cdot \frac{4}{8} \varphi_1^{n-3}$ is also $3 \cdot \frac{4}{16} \varphi^{n-4}$, but increases in triplicate ratio, and is reduced to synchronous measure by the divisor 3.

7. The generator $4 \varphi^{n-3}$ increases in triplicate ratio, and therefore its generatant φ^{n-4} increases in quadruplicate ratio: the successive effect of this generatant, in its function as generator, will necessarily be m''' times its synchronous effect; which authorizes the multiplication of the first of the following equations by m'''.

$$\tfrac{1}{2}' \cdot 1_t' + \tfrac{1}{2}'' \cdot 1_t' = 1_t' :$$

$$\tfrac{1}{16} \varphi_1^{n-4} + \tfrac{4}{16} \varphi_1^{n-4} + \tfrac{6}{16} \varphi^{n-4} + \tfrac{4}{16} \varphi_2^{n-4} + \tfrac{1}{16} \varphi_2^{n-4} = 1 \varphi^{n-4};$$

$$\frac{m'''}{16} \varphi_1^{n-4} + \frac{4m'''}{16} \varphi_1^{n-4} + \frac{6m'''}{16} \varphi^{n-4} + \frac{4m'''}{16} \varphi_2^{n-4} + \frac{m'''}{16} \varphi_2^{n-4} = m''' \varphi^{n-4}.$$

The force $\frac{m'''}{16} \varphi_1^{n-4}$ increases in quadruplicate ratio from 0, being of the fifth rank from its constant generator $1 \varphi^n$: its successive unit measure of effect becomes $\frac{m'''}{32} \varphi^{n-5}$ for the semi-unit of time $\frac{1}{2}'' \cdot 1_t'$, leaving $\frac{1}{32} \varphi^{n-5}$ for its synchronous measure in the time $\frac{1}{2}' \cdot 1_t'$, which is also the synchronous measure of the effect of $\frac{m'''}{16} \varphi_2^{n-4}$ for the time $\frac{1}{2}'' \cdot 1_t'$.

The force $\frac{4m'''}{16} \varphi_1^{n-4}$ increases in triplicate ratio, being of the fourth rank from its constant generator $\frac{2}{2} \varphi_1^{n-1}$: its successive semi-unit of

effect $\frac{4\,m'''}{32}\,\varphi^{n-5}$ increases quadruplicately, and must be divided by 4 to get $\frac{4}{4}\cdot\frac{m'''}{32}\,\varphi^{n-5}$ its synchronous effect for the time $\frac{1}{2}''.\,1_t'$.

The force $\frac{6\,m'''}{16}\,\varphi^{n-4}$ increases in duplicate ratio, being of the third rank from the constant generator $\frac{3}{4}\,\varphi_1^{n-3}$: its successive semi-unit effect $\frac{6\,m'''}{32}\,\varphi^{n-5}$ increases triplicately, and must be divided by 3 to get $\frac{6}{3}\cdot\frac{m'''}{32}\,\varphi^{n-5}$ its sychronous effect for the time $\frac{1}{2}''.\,1_t'$.

The force $\frac{4\,m'''}{16}\,\varphi_2^{n-4}$ increases in simple ratio, being of the second rank from the constant generator $\frac{4}{8}\,\varphi_1^{n-3}$: its successive semi-unit of effect $\frac{4\,m'''}{32}\,\varphi^{n-5}$ increases duplicately, and must be divided by 2 to get $\frac{4}{2}\cdot\frac{m'''}{32}\,\varphi^{n-5}$ its synchronous measure for the time $\frac{1}{2}''.\,1_t'$.

The collected results constitute the equation

$$\left(\tfrac{1}{32}+\tfrac{4}{4}\cdot\frac{m'''}{32}+\tfrac{6}{3}\cdot\frac{m'''}{32}+\tfrac{4}{2}\cdot\frac{m'''}{32}+\tfrac{1}{1}\cdot\frac{m'''}{32}+\tfrac{1}{32}\right)\varphi^{n-5}=m'''\,\varphi^{n-5},$$

which gives $m'''=5$, and converts the last preceding equation into

$$\tfrac{1}{2}'.\,1_t'+\tfrac{1}{2}''.\,1_t'=1_t':$$

$$\tfrac{5}{16}\,\varphi_1^{n-4}+\tfrac{20}{16}\,\varphi_1^{n-4}+\tfrac{30}{16}\,\varphi^{n-4}+\tfrac{20}{16}\,\varphi_2^{n-4}+\tfrac{5}{16}\,\varphi_2^{n-4}=5\,\varphi^{n-4}.$$

Thus the successive and synchronous measures of a force of the fifth rank are mutually convertible by the factor 5.

8. So much may suffice to elucidate the genetical process occurring during the first unit of time; which process may be prolonged downwards to the nth rank, terminating with $\varphi^{n-n}=\varphi^0$, the force of the order zero which is no generator.

9. For $n=4$, the two series of synchronous and successive unit measures of the generated forces, including the primitive generator, are

$$1\,\varphi^{\mathrm{IV}},\ 1\,\varphi''',\ 1\,\varphi'',\ 1\,\varphi',\ 1\,\varphi^0,$$

and

$$1\,\varphi^{\mathrm{IV}},\ 2\,\varphi''',\ 3\,\varphi'',\ 4\,\varphi',\ 1\,\varphi^0;$$

each term of the first series being the measure of the generated effect confined throughout to the first unit of time, while the coefficient of each term of the second series expresses the relation between the effects of its efficient in the times $1''_t$ and 1_t. Thus, while the synchronous effect of the force $3\varphi''$ for the time 1_t is $1\varphi'$ referred to synchronous effect of the latter force, it is $4\varphi'$ in successive measure, and the successive effect of the same force $3\varphi''$ is $3 \cdot 4\varphi'$ for the time $1''_t$.

The genetical operations for the first two units of time are as follows: —

$1'_t$.	$1''_t$.
$1\varphi^{IV} = 1\varphi^{IV}$:	$\frac{1.2}{1}\varphi''', \frac{1.2.3}{1.2}\varphi'', \frac{1.2.3.4}{1.2.3}\varphi', \frac{1.2.3.4.1}{1.2.3.4}\varphi^0 = 1\varphi^0.$
$1\varphi''' = 2\varphi'''$:	$\frac{2.3}{1}\varphi'', \frac{2.3.4}{1.2}\varphi', \frac{2.3.4.1}{1.2.3}\varphi^0 = 4\varphi^0.$
$1\varphi'' = 3\varphi''$:	$\frac{3.4}{1}\varphi', \frac{3.4.1}{1.2}\varphi^0 = 6\varphi^0.$
$1\varphi' = 4\varphi'$:	$\frac{4.1}{1}\varphi^0 = 4\varphi^0.$
$1\varphi^0 = 1\varphi^0.$	

The final effect of the force $1\varphi^{IV}$ in two units of time is

$$(1 + 4 + 6 + 4 + 1)\varphi^0 = 16\varphi^0 = 2^4\varphi^0.$$

In the first left-hand column, each term below the first expresses the synchronous value of the effect generated by the immediately superior term of the second column (directly on the right of the equality signs), during the time $1'_t$; each term of the second column increasing from 0 to the value expressed by its coefficient, and being at the same time the successive value of the effect generated by its immediately superior term of the same column.

Passing to the right of the colon points, the primitive force $1\varphi^{IV}$ generates in the second unit of time $1''_t$, a series equal to that of $1'_t$, closing with $\frac{1.2.3.4.1}{1.2.3.4}\varphi^0 = 1\varphi^0$.

The force $2\,\varphi'''$, now become a constant generator, having generated $3\,\varphi''$ while increasing uniformly during $1'_t$, will generate $2\,.\,3\,\varphi''$ in the time $1''_t$: this force $2\,.\,3\,\varphi''$ increases uniformly during $1''_t$, and therefore its successive measure $2\,.\,3\,.\,4\,\varphi'$ must be divided by 2 to get its synchronous value $\frac{2\,.\,3\,.\,4}{1\,.\,2}\,\varphi'$; and since the force $\frac{2\,.\,3\,.\,4}{1\,.\,2}\,\varphi'$ increases duplicately, and is of the first order, its successive effect, $\frac{2\,.\,3\,.\,4\,.\,1}{1\,.\,2}\,\varphi^0$, has no successive value, but must nevertheless be divided by 3 to get its synchronous value, $\frac{2\,.\,3\,.\,4\,.\,1}{1\,.\,2\,.\,3}\,\varphi^0 = 4\,\varphi^0$.

The force $3\,\varphi''$, now become a constant generator, and having generated $4\,\varphi'$ while increasing duplicately during $1'_t$, will generate $3\,.\,4\,\varphi'$ in the time $1''_t$: this force $3\,.\,4\,\varphi'$ increases uniformly during $1''_t$; and as it is of the first order, its successive effect, $3\,.\,4\,.\,1\,\varphi^0$, has no successive value, but must be divided by 2 to fetch its synchronous value $\frac{3\,.\,4\,.\,1}{1\,.\,2}\,\varphi^0 = 6\,\varphi^0$.

Finally, the force $4\,\varphi'$, now become a constant generator, and being of the first order, its successive effect, $4\,.\,1\,\varphi$, contains no element of increase, or is uniform and without successive value, and therefore has but the divisor 1 to bring its synchronous value $\frac{4\,.\,1}{1.}\,\varphi^0 = 4\,\varphi^0$ for the time $1''_t$.

10. If, then, a unit force $1\,\varphi^{IV}$ of the fourth order be applied to a unit of mass 1_μ during a unit of time 1_t, at the expiration of the first unit interval $1'_t$ the mass 1_μ has described a unit of distance $1_l = 1\,\varphi^0$ the unit force of the order zero, the phenomenal measure of the primitive force in and for the first unit of time: there is in the same time $1'_t$ also generated the velocity $4\,.\,1_\lambda$ or force $4\,\varphi'$ of the first order, which will, by uniform motion, carry the unit mass 1_μ the distance $4\,.\,1_l$ in the next unit of time $1''_t$; also the forces $2\,\varphi'''$ and $3\,\varphi''$, which each generate the velocity $12\,.\,1_\lambda$ in the time $1''_t$, but under different rates of acceleration, such as suffice to carry a unit mass respectively the distances $4\,.\,1_l$ and $6\,.\,1_l$ in the time $1''_t$ of their genesis; to these amounts add the unit of distance again gen-

erated by the primitive force $1\,\varphi^{IV}$ in the second unit of time, and there is generated the distance 1_t in the first unit of time; and in the second unit of time, the distance $4 \cdot 1_t$ by the velocity in existence at the beginning of this interval, and $11 \cdot 1_t$ by the forces in operation; making $15 \cdot 1_t$ for the time $1''_t$, and in all $16 \cdot 1_t = (1+1)^4\, 1_t$ for the whole time $2 \cdot 1_t$.

11. As the interval of time is arbitrary, it may be taken equal to $\psi\, 1_t$ or $h\, 1_t$, and so achieve the transition from $(1+1)^n$ to ψ^n and $(\psi + h)^n$, etc.

As the force $1\,\varphi^{IV}$ generates $2\,\varphi'''$ uniformly in the time 1_t, it will generate $2\,\psi\,\varphi'''$ uniformly in the time $\psi\, 1_t$.

As the force $2\,\varphi'''$ generates $2 \cdot 3\,\varphi''$ uniformly in the time $1''_t$ and $3\,\varphi''$ duplicately in the preceding equal time $1'_t$, so the force $2\,\psi\,\varphi'''$ will generate $2 \cdot 3\,\psi^2\,\varphi''$ uniformly in the time $\psi''\, 1_t$ and $3\,\psi^2\,\varphi''$ duplicately in the preceding equal time $\psi'\, 1_t$.

As the force $3\,\varphi''$ generates $3 \cdot 4\,\varphi'$ uniformly in the time $1''_t$ and $4\,\varphi'$ triplicately in the time $1'_t$, so the force $3\,\psi^2\,\varphi''$ will generate $3 \cdot 4\,\psi^3\,\varphi'$ uniformly in the time $\psi''\, 1_t$ and $4\,\psi^3\,\varphi'$ triplicately in the preceding equal time $\psi'\, 1_t$.

As the force $4\,\varphi'$ generates $4\,\varphi^0$ uniformly in the time $1''_t$ and $1\,\varphi^0$ quadruplicately in the time $1'_t$, so the force $4\,\psi^3\,\varphi'$ will generate $4\,\psi^4\,\varphi^0$ uniformly in the time $w''\, 1_t$ and $1\,\psi^4\,\varphi^0$ quadruplicately in the preceding equal time $\psi'\, 1_t$.

[To be Continued.]

Mathematical Monthly Notices.

Cambridge Senate-House Problems and Riders for the Year 1860; WITH SOLUTIONS. By the REV. H. W. WATSON, M. A., late Fellow of Trinity College; and E. J. ROUTH, M. A., Fellow and Assistant Tutor of St. Peter's College, Cambridge, and Examiner in the University of London. Cambridge: Macmillan & Co., and 23 Henrietta Street, Covent Garden, London. 1860. pp. 198. 3 plates containing 55 cuts.

The Moderators for the year 1860 are the Editors of the volume; the Examiners are PERCIVAL FROST, M. A., St. John's College, and NORMAN MACLEOD FERRERS, M. A., Gonville

and Caius College. The examples alone, collected and arranged in this series of volumes, form a valuable annual contribution to mathematical science; but their value, especially to students, is greatly increased, when, as in this volume, the problems are accompanied with solutions by those who proposed them. We earnestly invite the attention of teachers and students to this volume, not only for what it contains, but also as indicating the standard in the great mathematical University.

A Treatise on Plane Co-ordinate Geometry, as applied to the Straight Line and the Conic Sections, With numerous Examples. By I. TODHUNTER, M. A., Fellow and Assistant Tutor of St. John's College, Cambridge. Second edition revised. Cambridge: Macmillan & Co. 1858. pp. 316.

This is one of a series of eight or nine volumes, prepared by the author for the use of schools and colleges. Great care has been taken in the methods of demonstration and in the explanations to adapt them to the wants of beginners. There is no branch of mathematics in which we discover such important changes and improvements in the text-books, in use in Great Britain, as in Analytic Geometry. We have already called the attention of our readers to two valuable works upon this subject (SALMON'S and PUCKLE'S); and now we are able to say, that, after a careful examination, we can heartily commend the one before us. Besides the usual propositions upon the straight line and conics, it contains an exposition of many of the new processes more recently introduced into elementary works, leaving no doubt of their having secured a permanent place in elementary instruction. Each chapter is followed by a large number of appropriate examples, and at the end of the volume we find all the answers collected together, with hints for the solution of many of those which appear more difficult.

Algebra for the Use of Colleges and Schools. With numerous Examples. By I. TODHUNTER, M. A., Fellow of St. John's College, Cambridge. Second edition revised. Macmillan & Co., Cambridge, and 23 Henrietta Street, Covent Garden, London. 1860. pp. 516.

We have examined this work with unusual interest, because it is the first recent work upon the subject, used abroad, which has fallen into our hands, thus enabling us to compare it with works at home filling the same place in our system of instruction. We notice, first, that the general theory of equations is entirely omitted, and also the solution of equations of all degrees above the second; nor does the work include any discussion making it necessary to anticipate, by definition or otherwise, even the elements of the higher parts of the course; and second, the introduction of many subjects not usually treated in the Elements of Algebra. An enumeration of some of them will show the extent and direction of this innovation; a few subjects usually treated being included on account of the clearness and fulness with which they are developed. Theory of *Quadratic Expressions* in addition to Quadratic Equations; Imaginary Expressions; Ratio introductory to Proportion; Variation; Scales of Notation; Harmonical Progression; Mathematical Induction; Multinomial Theorem; Logarithms, with Exponential and Logarithmic Series; Convergency and Divergency of Series; Indeterminate Equations of the First and Second Degrees; Partial Fractions and Indeterminate Coefficients; Recurring Series; Summation of Series; Inequalities; Theory of Numbers; Probability; Miscellaneous Theorems. All these subjects are treated in a very plain and familiar style; and the whole volume possesses many of the best traits of a good text-book. The author says: "Students should be encouraged to examine carefully the language of the book they are using, so that they may ascertain its meaning, or be able to point out exactly where their difficulties arise. The language, therefore, ought to be simple and precise; and it is essential that apparent conciseness should not be gained at the expense of clearness." Students, who are pursuing their mathematical studies without the aid of a teacher, will find this work well adapted to their wants. It contains about *sixteen hundred and fifty* examples, the best collection for the school-room we have ever seen.

University Algebra; embracing a Logical Development of the Science, with numerous graded Examples. By CHARLES DAVIES, LL. D., Professor of Mathematics in Columbia College. New York: A. S. Barnes and Burr, 51 and 53 John Street. 1860. pp. 320.

Our acquaintance with all of the author's more advanced books, both as a student and teacher, and our conviction that they have accomplished a vast good to the cause of sound education in this country, makes us feel a deep interest in every new addition to the series. A careful examination of the work before us shows that it is intended to occupy a place intermediate between the first elements of algebra and advanced treatises, a fact which the term *University* might not indicate to all; and, that it is one of the most carefully prepared works of the author's series, possessing so many of the requisites of a good text-book, as to insure it an extensive introduction as soon as its merits shall become known. Besides the fact that the definitions, demonstrations, and explanations are all expressed with great clearness, and in that happy mean between extreme conciseness and prolixity, which constitutes a model didactic style in mathematical writing, we beg to note the following special points. The demonstration of the Binomial Theorem for positive integral powers is well adapted for beginners. A few of the successive powers of a binomial are found by actual multiplication from which the general laws of exponents and coefficients are inferred; these laws are then assumed for the mth power, and, shown to hold for the $(m+1)$st power, a species of proof known as mathematical *induction*.

The case of the Problem of the Lights discussed in the Monthly, Vol. I. pp. 160, 178, and also in the present number, is thus disposed of: "The hypothesis of $c = 0$, places the two lights at the same point, in which case they form one and the same light, whose intensity is equal to the sum of their intensities taken separately. The conditions of the problem involve the necessity of *two lights*, and the equation of the problem is found under this hypothesis. This equation ought not, therefore, to respond to the case of a single light. For the interpretation of the results obtained from an equation can only give the cases which fall under the hypothesis. The hypothesis of two lights, and the hypothesis of a single light, are not connected by any law which affords a common equation of condition. Hence, the results obtained on the supposition $c = 0$ do not belong to the problem."

It certainly is possible, mathematically, to suppose that two lights of unequal intensities may occupy the same point and still be regarded as separate and distinct lights; and the question may properly be asked whether two such lights, so situated and regarded, can illuminate any point in space by the same amount. It seems to us, therefore, that it would have been better to have shown, that, under this hypothesis, the condition that a point should be equally illuminated by the two lights becomes impossible. This would have obviated a rather doubtful explanation of the difficulty.

The chapter on the General Theory of Equations is brief, but as full as is consistent with the plan of the work. We notice that derived equations are given by simple definition, no attempt at an exposition of the principles involved being made. This seems to suggest whether it is not better to postpone the general theory of equations till later in the course, after the student shall have acquired, at least the elements of the calculus. The great fault with many of our text-books is, that they encroach upon each other's ground; and thus consume too much of the student's time by simple repetition.

After giving the form of the general equation of the nth degree, containing but one unknown quantity, with positive or negative, real or imaginary coefficients, the author says: "It is evident that this equation has at least one root, for if the first member is equal to 0, it must be so for some value of x, either real or imaginary, and this value of x is, by definition, a root." The only difficulty in this disposition of a proposition, which has lately received much attention from eminent mathematicians, is, that it assumes the point to be proved. *If a function of x equals*

zero, as a matter of course it must be so for some value of x, either real or imaginary. But is there necessarily some value of x of the form of $a+b\sqrt{-1}$ which will make the function zero? This is the question.

Astronomical Notices. — No. 21. Edited by DR. F. BRÜNNOW, Professor of Astronomy and Director of the Observatory of Michigan University, Ann Arbor. Ann Arbor, October 11th, 1860.

The contents of this number are, a letter from R. F. BOND, ESQ., on a New Clock-Escapement, with a cut; a letter from COMMANDER MAURY, announcing the discovery of a new Asteroid, September 14th, by MR. FERGUSON, with Observations of Leucothea, Comet III., 1860, and Solar Eclipse, July 18th; Co-ordinates of Spots visible on the Sun, July 18th, and Observation of the Solar Eclipse by DR. C. H. F. PETERS, Director of the Observatory of Hamilton College.

A Collection of Problems and Examples in Mathematics; selected from the Jesus College Examination Papers, with Answers. — By H. A. MORGAN, M. A., Sadlerian and Mathematical Lecturer of Jesus College, Cambridge. Cambridge: Macmillan & Co. 1858. pp. 190.

This is a collection of simple examples for the use of students, arranged as follows, with the number under each head: Euclid, 55; Conic Sections, Parabola, 27; Ellipse, 29; Hyperbola, 6; Algebra, 66; Theory of Equations, 23; Trigonometry, 96; Analytical Geometry of Two Dimensions, 94; Differential and Integral Calculus, 70; Differential Equations, 15; Geometry of Three Dimensions, 26; Elementary Statics, 61; Analytical Statics, 47; Elementary Dynamics, 57; Newton, Books I., II., III., 54; Dynamics of a Particle, 33; Rigid Dynamics, 32; Hydrostatics, 57; Optics, 72· Spherical Trigonometry, and Astronomy, 32· Answers to the Questions.

Editorial Items.

PRIZE SOLUTIONS. — Owing to the delay in the publication of the September No., Vol. II., and October No., Vol. III., one month additional will be given for the solutions of the problems in these numbers. We trust that students will continue their interest in this feature of the Monthly, and that teachers will encourage their pupils to send us solutions.

LAPLACE'S *Autograph.* — It will be seen that there is no doubt about the genuineness of this Autograph. Very little of LAPLACE'S manuscript is still in existence, and we thank MR. WATSON for enabling us to present copies of this interesting note to our readers.

MATHEMATICAL TEXT-BOOKS. — We propose to give more attention than formerly to Elementary Text-books in Mathematics, Natural Philosophy, &c. A proper proportion of each number will be thus devoted.

ERRATA. — On page 419, Vol. II., line 10, for x^2 read nx^2; p. 421, line 7, read $R=\frac{r}{n}\sqrt{1-n}$, and on line 14, read $R=\sqrt{A^2-B^2}$.

MR. EVANS has called our attention to the misstatement of Problem V. in the September Number. It should read, "Prove that a chord joining one pair of these points will meet two other such chords each at a point on an asymptote, and be *in the case of the circle* perpendicular to one of them."

THE

MATHEMATICAL MONTHLY.

Vol. III. . . . DECEMBER, 1860. . . . No. III.

PRIZE PROBLEMS FOR STUDENTS.

I. From AC, the diagonal of a square $ABCD$, cut off AE equal to one fourth of AC, and join BE, DE. Show that the figure $BADE$ equals twice the square on AE.

II. Two toothed wheels work against each other. Show that, if the number of teeth in one be prime to that in the other, before two teeth, which have once been in contact, come in contact again, every tooth of the one wheel will have been in contact with every tooth of the other.

III. Eliminate φ from the equations

$$x\cos(\varphi+\alpha)+y\sin(\varphi+\alpha)=a\sin 2\varphi,$$
$$y\cos(\varphi+\alpha)+x\sin(\varphi+\alpha)=2a\cos 2\varphi,$$

showing that

$$(x\sin\alpha-y\cos\alpha)^{\frac{2}{3}}+(y\sin\alpha+x\cos\alpha)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}.$$

IV. Show that $\frac{1}{4}m^2$ is the area of the greatest triangle which can be formed with the lines a, b, c, subject to the condition $a^3+b^3+c^3=3m^3$.

V. The equation of a family of ellipses is $ax^2+by^2=1$, when $a-b=c$, a constant. Show that the curve, which cuts all the individuals at an angle whose tangent is $\frac{y}{x}$, is $x=Ce^{-\frac{cy^2}{2}}$.

Solutions of these problems must be received by February 1, 1861.

NOTES AND QUERIES.

1. *Perfect Squares.* — It is a well-known fact that every perfect square, when represented by the Arabic or Hindoo method, must end in 0, 1, 4, 5, 6, or 9. This fact enables us to decide, at a glance, that all numbers ending in 2, 3, 7, or 8 are not perfect squares; but it does not aid us in deciding which of the numbers ending in 0, 1, 4, 5, 6, or 9 are perfect squares, for the converse of the proposition stated above is not true. To assist, to some extent, in removing this difficulty, I submit the following discussion.

PROPOSITION. — The last two figures of the square of any integer must correspond to the last two figures of the square of some number in the series 1, 2, . . 99. The endings of the squares of the first twenty-five numbers form the series 01, 04, 09, 16, 25, 36, 49, 64, 81, 00, 21, 44, 69, 96, 25, 51, 89, 24, 61, 00, 41, 84, 29, 71, 24. This series also represents the endings of the squares of the numbers from 51 to 75, inclusive. The series, taken backwards, represents the endings of the numbers from 25 to 49, and from 75 to 99, inclusive. This shows that the squares of the numbers equally above and below 25 and 75 end in the same two figures. Omitting the like endings in the above series, and arranging in order, we have 00, 01, 04, 09, 16, 21, 24, 25, 29, 36, 41, 44, 49, 56, 61, 64, 69, 76, 81, 84, 89, 96. These endings indicate the following facts: Every perfect square ending in 1, 4, or 9 must have before the last figure one of the even numbers 0, 2, 4, 6, or 8; ending in 6, must be preceded by one of the odd numbers 1, 3, 5, 7, or 9; ending in 5, by 2; and ending in 0, by 0. — Prof. W. D. HENKLE, Normal School, Lebanon, Ohio.

2. *Problem.* — Prove that, when x is a whole number and greater than one, $x^3 - x$ is divisible by 6.

Suppose that $x^3 - x$ is divisible by 6 when $x = p$, then will it be divisible by 6 when $x = p + 1$. For, letting $x = p + 1$, we have

$$x^3 - x = p^3 + 3p^2 + 3p + 1 - (p+1) = p^3 + 3p^2 + 2p = p^3 - p + 3p^2 + 3p.$$

Now $p^3 - p$ is divisible by 6, by hypothesis; and if we can prove that $3p^2 + 3p$ is also divisible by 6, then it will follow that $x^3 - x$ is divisible by 6 when $x = p + 1$. But to prove that $3p^2 + 3p$ is divisible by 6, we must prove that $p^2 + p$ is divisible by 2. Suppose that $p^2 + p$ is divisible by 2 when $p = r$, then will it be divisible by 2 when $p = r + 1$. For, putting $p = r + 1$, we get

$$p^2 + p = r^2 + 2r + 1 + r + 1 = r^2 + 3r + 2 = r^2 + r + 2r + 2.$$

But $r^2 + r$ is divisible by 2 by hypothesis, and $2r + 2$ is also divisible by 2. Now we know that $p^2 + p$ is divisible by 2 when $p = 1$, $\therefore$ it must be when $p = 2$, 3, 4, &c.; and therefore if $x^3 - x$ is divisible by 6 when $x = p$, it will be when $x = p + 1$. But we know by inspection that it is divisible when $x = 2$; $\therefore$ it will be when $x = 3$, 4, 5, &c. — JOHN SCOTT, Jr., Night High School, Cincinnati, Ohio.

3. *Note on the Prize Solution of Problem III.*, p. 25. — As there has been a slight typographical error each time the formulas have been printed, I send them in their correct and most reduced form. As m and n exchange values in the two solids, the formulas take very nearly the same forms, one having cosine where the other has the square of the cotangent, and 3 in the place of 2. The last forms given below are adapted to logarithmic computation.

$$\text{Vol. Dodec.} = \frac{5e^3 \cot^2\beta}{2\sqrt{(4\sin^2\beta - 1)}} = e^3\left(\frac{15 + 7\sqrt{5}}{4}\right) = \frac{5e^3 \tan^2 54^\circ}{4\sqrt{(\sin 6^\circ \cos 24^\circ)}}.$$

$$\text{Vol. Icos.} = \frac{5e^3 \cos\beta}{3\sqrt{(3 - 4\cos^2\beta)}} = \frac{5e^3 \cos\beta}{3\sqrt{(4\sin^2\beta - 1)}}$$

$$= e^3\left(\frac{15 + 5\sqrt{5}}{12}\right) = \frac{5e^3 \sin 54^\circ}{6\sqrt{(\sin 6^\circ \cos 24^\circ)}}.$$

— PROF. D. W. HOYT, New Hampton Institution, Fairfax, Vt.

4. *Correction of the Method of Computing Six per Cent Interest for Days.* — The usual method of computing interest at six per cent, for a given

number of days, that is, "Divide the principal by 6,000 and multiply the quotient by the given number of days," is not of course accurate when large numbers are involved; but the following development, p, denoting the principal, and d the days,

$$\frac{0.06\,p\,d}{365.25} = \frac{p\,d}{6000}\,\{1 + \tfrac{1}{70}\,[1 + \tfrac{1}{60}\,(1 + \tfrac{1}{24})]\},$$

shows that the correction is almost exactly $\frac{1}{70}$ of the first result; and if this is not sufficient, the second correction is $\frac{1}{60}$ of the first. The third correction, $\frac{1}{24}$ of the second, will never be needed, and is not quite so convenient to apply mentally as the others, but it is accurate to 0.000001 of its value. It will be noticed that these corrections are all additive. All except the first, however, are more curious than useful. — B.

5. *Solution of Problem V., No. IX., Vol. II.* — Two great circles are drawn at random on a sphere. What is the probability that their mutual inclination, taken less than 90°, will be contained between any given limits, as $n°$ and $m°$?

1st. The mutual inclination of the two circles will be equal to the angular distance of their poles.

2d. The pole of either great circle is just as likely to fall in one position as in any other; hence the probability that it will fall in any given portion of the surface of the sphere is proportional to the superficial area of that portion.

3d. In order that the angular distance of the poles may be between the limits $n°$ and $m°$, one pole must fall within the zone contained between two small circles at the respective distances of $n°$ and $m°$ from the other pole.

4th. Hence the probability required will be equal to the area of this zone divided by the area of the hemisphere, or to $\cos n° - \cos m°$. Laplace, in considering the inclinations of the orbits of the planets

and comets, in the *Théorie Analytique des Probabilités*, fell into the error of supposing that this probability would be equal to $\frac{m^\circ - n^\circ}{90^\circ}$; or that all inclinations would be equally probable. — S. NEWCOMB, Cambridge, Mass.

6. *Note on the Powers of Binomials.* — Find the fifth power of the binomial $\frac{1}{2}x - 3y^2$.

$1 \quad +5 \quad +10 \quad +10 \quad +5 \quad +1,$	Coefficients.
$\frac{1}{32}x^5 + \frac{1}{16}x^4 + \frac{1}{8}x^3 + \frac{1}{4}x^2 + \frac{1}{2}x \quad +1,$	Power of $\frac{1}{2}x$.
$1 \quad -3y^2 \quad +9y^4 - 27y^6 + 81y^8 - 243y^{10},$	Powers of $-3y^2$.

$$\frac{1}{32}x^5 - \frac{15}{16}x^4y^2 + \frac{45}{4}x^3y^4 - \frac{135}{2}x^2y^6 + \frac{405}{2}xy^8 - 243y^{10} = (\tfrac{1}{2}x - 3y^2)^5.$$

This simple method of writing powers of complicated binomials is sufficiently obvious. Writing the different lines, and then multiplying together the terms which stand in the same column, so divides up the process, that the whole is made plain and easy to the student. — DAVIES'S University Algebra.

7. *The arithmetic mean of any number of positive quantities is greater than the geometric mean;* that is,

$$\frac{a+b+c+d\ldots+k}{n} > (a\,b\,c\,d\ldots k)^{\frac{1}{n}}.$$

Let P denote $(a\,b\,c\,d\ldots k)^{\frac{1}{n}}$, and Q denote $\frac{a+b+c+d\ldots k}{n}$. Suppose a and b respectively the greatest and least of the n quantities $a, b, c, d, \ldots k$; and let $a_1 = b_1 = \frac{1}{2}(a+b)$, and let $P_1 = (a_1 b_1 c d \ldots k)^{\frac{1}{n}}$; then, since $a_1 b_1 > a\,b$, we have $P_1 > P$. Next, if the factors in P_1 be not all equal, remove the greatest and least of them, and put in their places two new factors, each equal to half the sum of those removed; let P_2 denote the new geometrical mean; then $P_2 > P_1$. If we proceed in this way, we obtain a series, $P, P_1, P_2, P_3, \ldots P_r$, each term of which is greater than the preceding term; and by

taking r large enough, we may have the factors of P_r as nearly equal as we please; thus, when r is large enough, we may consider $P_r = Q$, therefore P is less than Q. — TODHUNTER'S Algebra.

8. *Note on the Successive Derivatives of tan* φ. — We have

$$D \tan \varphi = D \frac{\sin \varphi}{\cos \varphi} = \frac{\cos^2 \varphi + \sin^2 \varphi}{\cos^2 \varphi} = \frac{1}{\cos \varphi^2}.$$

$$D^2 \tan \varphi = -\frac{2 D \cos \varphi}{\cos^3 \varphi} = \frac{2 \sin \varphi}{\cos^3 \varphi} = \frac{2 \tan \varphi}{\cos^2 \varphi} = 2 \tan \varphi \; D \tan \varphi.$$

For brevity, let us adopt the notation $\tan \varphi = D^0$, $D \tan \varphi = D^1$, $D^2 \tan \varphi = D^2$, &c. Then

$$D^2 \tan \varphi = 2 D^1 D^0,$$

$$D^3 \tan \varphi = 2 D^2 D^0 + 2 (D^1)^2,$$

$$D^4 \tan \varphi = 2 D^3 D^0 + 2 D^2 D^1 + 4 D^2 D^1 = 2 D^3 D^0 + 6 D^2 D^1,$$

$$\begin{aligned} D^5 \tan \varphi &= 2 D^4 D^0 + 2 D^3 D^1 + 6 D^3 D^1 + 6 (D^2)^2 \\ &= 2 D^4 D^0 + 8 D^3 D^1 + 6 (D^2)^2, \end{aligned}$$

$$\begin{aligned} D^6 \tan \varphi &= 2 D^5 D^0 + 2 D^4 D^1 + 8 D^4 D^1 + 8 D^3 D^2 + 12 D^3 D^2 \\ &= 2 D^5 D^0 + 10 D^4 D^1 + 20 D^3 D^2. \end{aligned}$$

In general,

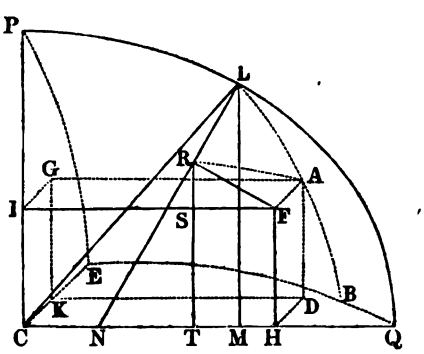
$$\begin{aligned} D^n \tan \varphi &= (D^1 + D^0)^{n-1} \\ &= D^{n-1} D^0 + (n-1) D^{n-2} D^1 + \frac{(n-1)(n-2)}{1 \cdot 2} D^{n-3} D^2 + \frac{(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3} D^{n-4} D^3 + \&c. \end{aligned}$$

In this series, terms equally distant from the extremes are equal, and may be added together. These very curious and interesting forms are found in PEIRCE'S *Curves and Functions*, Vol. I.

THE EARTH CONSIDERED AS A SPHEROID OF REVOLUTION.—GEODETIC FORMULAS.

By Dr. R. C. Matthewson, U. S. Deputy Surveyor, San Francisco.

Let $CEPQ$ represent one eighth of the spheroid, bounded by the convex surface EPQ, the plane of the equator EQC, and any other two planes, CPE and CPQ, at right angles to each other, passing through the pole P, and the centre C. On the plane CPQ,* from any point L in the meridian PQ, draw the radius LC, the normal LN, and the ordinate LM. Let the meridional plane CPQ be intersected in L by the normal plane LNB, making with it any angle BLQ, BL being the intersection with the surface. Then the curve BL, as well as PQ, will be the arc of an ellipse. In BL take any point A, and from it let fall the perpendiculars AD, AF, and AG respectively upon the planes CQE, CPE, and CPQ, and complete the parallelograms AH, AI, HK, and IK. On the plane CPQ draw FR at right angles to LN, and RT at right angles to CQ, intersecting FI in S, and join AR.



Put $CM = x$ and $LM = y$, the rectangular co-ordinates of the point L of the meridional ellipse PQ, C being the origin; $NR = x'$ and $AR = y'$, the rectangular co-ordinates of the point A of the normal ellipse BL, N being the origin; $AD = x''$, $AF = y''$, and $AG = z''$, the rectangular co-ordinates of the point A of the spheroidal surface EPQ, C being the origin; $CQ = a$, the equatorial

* For sake of distinction, the lines upon the plane CPQ are continuous, while those not upon it are dotted.

radius; $CP = b = a(1-e^2)^{\frac{1}{2}}$, the semi-axis, e being the eccentricity; $LN = n$, the normal; $MN = s$, the subnormal; $CN = m$, the adnormal; $CL = r$, the radius of the spheroid at the point L; the angle $LNM = l$, the geographical latitude of the point L; the angle $LCM = l'$, the geocentric latitude of the same point · and the angle $ALQ = z$, the azimuth of the point A from L.

Then the equation of the meridional ellipse PQ is

$$a^2y^2 + b^2x^2 = a^2b^2,$$

which, by eliminating b, becomes $y^2 = (1-e^2)(a^2-x^2)$. Differentiating this equation, we find the length of the subnormal,

$$s = y\frac{dy}{dx} = (1-e^2)\,x = n\cos l.$$

Hence $$x = \frac{n\cos l}{1-e^2}, \text{ and } y = n\sin l.$$

Substituting these values for x and y in the equation of the meridional ellipse, we have the normal, $n = \frac{a(1-e^2)}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$; and consequently the abscissa, $x = \frac{a\cos l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$; the ordinate, $y = \frac{a(1-e^2)\sin l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$;

the subnormal, $s = (1-e^2)\,x = \frac{a(1-e^2)\cos l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$;

the adnormal, $m = x - (1-e^2)\,x = e^2x = \frac{ae^2\cos l}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$;

the radius, $r = (x^2+y^2)^{\frac{1}{2}} = a\left(1 - \frac{e^2(1-e^2)\sin^2 l}{1-e^2\sin^2 l}\right)^{\frac{1}{2}}$;

and the geocentric latitude,

$$l' = \frac{s}{x}\tan l = \frac{(1-e^2)\,x}{x}\tan l = (1-e^2)\tan l.$$

Again, the equation of the spheroidal surface EPQ is

$$a^2x'^2 + b^2y'^2 + b^2z'^2 = a^2b^2,$$

or, by eliminating b, $x'^2 + (1-e^2)(y'^2 + z'^2 - a^2) = 0$. Now

in the right-angled triangle AFR we have the hypotenuse $AR = y'$, and the angle $ARF = z$, whence $AF = y' \sin z$ and $FR = y' \cos z$; in the right-angled triangle FRS we have the hypotenuse $FR = y' \cos z$ and the angle $FRS = l$, whence $FS = y' \sin l \cos z$ and $RS = y' \cos l \cos z$; and in the right-angled triangle NRT we have the hypotenuse $NR = x'$ and the angle $RNT = l$, whence $RT = x' \sin l$ and $NT = x' \cos l$. Hence,

$$x'' = AD = IC = RT - RS = x' \sin l - y' \cos l \cos z;$$

$$y'' = AF = CK = y' \sin z;$$

$$z'' = AG = CH = FS + NT + CN = y' \sin l \cos z + x' \cos l + m.$$

Now if we substitute the foregoing values for x'', y'', and z'' in the equation of the spheroidal surface EPQ, we shall find the equation of the normal ellipse BL to be

$$Ax'^2 + By'^2 + Cx'y' + Dx' + Ey' + F = 0,$$

in which $A = 1 - e^2 \cos^2 l$; $B = 1 - e^2 + e^2 \cos^2 l \cos^2 z$;

$$C = -2e^2 \sin l \cos l \cos z; \qquad D = \frac{2ae^2(1-e^2)\cos^2 l}{(1-e^2\sin^2 l)^{\frac{1}{2}}};$$

$$E = \frac{2ae^2(1-e^2)\sin l \cos l \cos z}{(1-e^2\sin^2 l)^{\frac{1}{2}}}; \quad \text{and} \quad F = -(1-e^2)(a^2 - m^2).$$

Differentiating the equation of the normal ellipse BL twice, and, in each of the differential coefficients, making $x' = n = \frac{a(1-e^2)}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$ and $y' = 0$, the values of the co-ordinates when the point A coincides with L, we have

$$\frac{dx'}{dy'} = 0, \qquad \text{and} \qquad \frac{d^2x'}{dy'^2} = -\frac{2D}{2Ax' + D}.$$

Substituting these values for $\frac{dx'}{dy'}$, and $\frac{d^2x'}{dy'^2}$ in the general expression for the radius of the osculating circle, and then substituting the above values of A, B, D, and x' in the result, we have

$$R_z = -\frac{\left(1+\frac{dx'^2}{dy'^2}\right)^{\frac{1}{2}}}{\frac{d^2x'}{dy'^2}} = \frac{Ax'+\frac{1}{2}D}{B} = \frac{a(1-e^2)}{(1-e^2+e^2\cos^2 l\cos^2 z)(1-e^2\sin^2 l)^{\frac{1}{2}}}$$

$$= \frac{a(1-e^2)}{(1-e^2\sin^2 z-e^2\sin^2 l\cos^2 z)(1-e^2\sin^2 l)^{\frac{1}{2}}}$$

$$= \frac{a(1-e^2)}{(\sin^2 z+\cos^2 z-e^2\sin^2 z-e^2\sin^2 l\cos^2 z)(1-e^2\sin^2 l)}$$

$$= \frac{a(1-e^2)}{(1-e^2)(1-e^2\sin^2 l)^{\frac{1}{2}}\sin^2 z+(1-e^2\sin^2 l)^{\frac{1}{2}}\cos^2 z},$$

which is the general expression for the radius of curvature of any normal ellipse on the spheroid making, at any latitude l, any angle z with the meridian.

When the normal ellipse BL coincides with the meridian, the angle $z = 0$, $\sin z = 0$, $\cos z = 1$, and R_z becomes $R_m = \frac{a(1-e^2)}{(1-e^2\sin^2 l)^{\frac{3}{2}}}$, which is the radius of curvature of the meridian at the latitude l. When the normal ellipse BL is at right angles to the meridian, the angle $z = 90°$, $\sin z = 1$, $\cos z = 0$, and R_z becomes $R_n = \frac{a}{(1-e^2\sin^2 l)^{\frac{1}{2}}}$, which is the radius of curvature of the perpendicular to the meridian at the latitude l. If $l = 0$, $\sin l = 0$, and we have $R_m = a(1-e^2)$ and $R_n = a$. If $l = 90°$, $\sin l = 1$, and we have $R_m = R_n = \frac{a}{(1-e^2)^{\frac{1}{2}}}$. In the former case the normal ellipse BL coincides with the equator, while in the latter it coincides with the meridian at the pole.

The radius of curvature of any normal ellipse on the spheroid, whether coinciding with the plane of the meridian, at right angles, or oblique to it, may also be obtained in the following manner. The radius of curvature of the meridian may be derived, at once, from the equation of the meridional ellipse $y^2 = (1-e^2)(a^2-x^2)$, by substituting the values of the first and second differential coefficients in the general expression for the radius of the osculating circle, and making the proper substitutions and eliminations. It will be found that

$$\frac{dy^2}{dx^2} = -\frac{(1-e^2)x}{a^2-x^2}, \qquad \text{and} \qquad \frac{d^2y}{dx^2} = -\frac{a^2(1-e^2)^{\frac{1}{2}}}{(a^2-x^2)^{\frac{3}{2}}},$$

whence $$R_m = \frac{(a^2 - e^2 x^2)^{\frac{1}{2}}}{a^2(1-e^2)} = \frac{a(1-e^2)}{(1-e^2\sin^2 l)^{\frac{1}{2}}}.$$

The radius of curvature of the perpendicular to the meridian is the normal LN produced to the axis of the spheroid, whence it is evident that $MN : CM :: LN : R_n$, or $S : x :: n : R_n$ and consequently

$$R_n = \frac{nx}{S} = \frac{nx}{(1-e^2)x} = \frac{n}{1-e^2} = \frac{a}{(1-e^2\sin^2 l)^{\frac{1}{2}}}.$$

It can then be easily proved that

$$R_z = \frac{R_m R_n}{R_m \sin^2 z + R_n \cos^2 z}* = \frac{a(1-e^2)}{(1-e^2)(1-e^2\sin^2 l)^{\frac{1}{2}}\sin^2 z + (1-e^2\sin^2 l)^{\frac{1}{2}}\cos^2 z}.$$

In order to find the length of a small arc δ° of a normal ellipse, in any position, on the surface of the spheroid, we have merely to multiply the radius of curvature at l, the middle latitude of δ°, by π, the ratio of the semi-circumference to the radius of a circle, and divide the product by $\frac{180^\circ}{\delta^\circ}$. If $\delta = 1^\circ$, we have

$$D_n = \frac{\pi}{180} R_n = \frac{\pi}{180} \cdot \frac{a}{(1-e^2\sin^2 l)^{\frac{1}{2}}} = \frac{\pi a}{180}(1-e^2\sin^2 l)^{-\frac{1}{2}},$$

which is the length of an arc of one degree of the perpendicular to the meridian at the mean latitude l;

$$D_m = \frac{\pi}{180} R_m = \frac{\pi}{180} \cdot \frac{a(1-e^2)}{(1-e^2\sin^2 l)^{\frac{1}{2}}} = \frac{\pi a(1-e^2)}{180}(1-e^2\sin^2 l)^{-\frac{1}{2}},$$

which is the length of an arc of one degree of the meridian at the mean latitude l; and

$$D_z = \frac{\pi}{180} R_z = \frac{\frac{\pi}{180} R_m R_n}{R_m \sin^2 z + R_n \cos^2 z},$$

which is the length of an arc of one degree making any angle z with the meridian, at the mean latitude l.

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* This Formula is wrong in Capt. T. J. Lee's Tables and Formulæ, Article XI. p. 34.

If, in the last expressions for the length of a degree of the perpendicular and meridian, we develop the second factors by the binomial theorem, we have

$$D_n = \frac{\pi a}{180}\left(1 + \tfrac{1}{2}e^2\sin^2 l + \tfrac{3}{8}e^4\sin^4 l + \tfrac{5}{16}e^6\sin^6 l + \tfrac{35}{128}e^8\sin^8 l + \&c.\right),$$

and

$$D_m = \frac{\pi a(1-e^2)}{180}\left(1 + \tfrac{3}{2}e^2\sin^2 l + \tfrac{15}{8}e^4\sin^4 l + \tfrac{35}{16}e^6\sin^6 l + \tfrac{315}{128}e^8\sin^8 l + \&c.\right).$$

Now $\pi = 3.141592654$, and, according to the data adopted by the Superintendent of the United States Coast Survey, $a = 6377397.156$ metres, and $e = 0.081696830$. Substituting these values for π, a, and e, in the developed expressions for the length of a degree of the perpendicular and meridian, we have, in powers of $\sin l$,

$$(1)\quad D_n = 111306.57808 + 371.45076\sin^2 l + 1.85940\sin^4 l + 0.01034\sin^6 l + 0.0006\sin^8 l.$$

$$(2)\quad D_m = 110563.67657 + 1106.91467\sin^2 l + 9.23495\sin^4 l + 0.07191\sin^6 l + 0.00054\sin^8 l.$$

But $\sin^2 l = 1 - \cos^2 l$; $\sin^4 l = 1 - 2\cos^2 l + \cos^4 l$;

$\sin^6 l = 1 - 3\cos^2 l + 3\cos^4 l - \cos^6 l$;

and $\sin^8 l = 1 - 4\cos^2 l + 6\cos^4 l - 4\cos^6 l + \cos^8 l$.

Substituting these values for the different powers of $\sin l$ in (1) and (2), we have, for the length of a degree of the perpendicular and meridian in powers of $\cos l$,

$$(3)\quad D_n = 111679.89864 - 375.20082\cos^2 l + 1.89079\cos^4 l - 0.01058\cos^6 l + 0.00006\cos^8 l.$$

$$(4)\quad D_m = 111679.89864 - 1125.60246\cos^2 l + 9.45392\cos^4 l - 0.07407\cos^6 l + 0.00054\cos^8 l.$$

Now $\sin^2 l = \frac{1}{2} - \frac{1}{2}\cos 2l$; $\sin^4 l = \frac{3}{8} - \frac{1}{2}\cos 2l + \frac{1}{8}\cos 4l$;

$\sin^6 l = \frac{5}{16} - \frac{15}{32}\cos 2l + \frac{3}{16}\cos 4l - \frac{1}{32}\cos 6l$;

and $\sin^8 l = \frac{35}{128} - \frac{7}{16}\cos 2l + \frac{7}{32}\cos 4l - \frac{1}{16}\cos 6l + \frac{1}{128}\cos 8l$. Substituting these values of the different powers of $\sin l$ in (1) and (2), or changing all the negative into positive signs, which changes the values to those of the powers of $\cos l$ instead of $\sin l$, and substituting in (3) and (4), we have, for the length of a degree of the perpendicular and meridian in multiples of $\cos l$,

(5) $D_n = 111493.00398 - 186.65995\cos 2l + 0.23438\cos 4l - 0.00033\cos 6l.$

(6) $D_m = 111120.61963 - 558.10875\cos 2l + 1.16797\cos 4l - 0.00228\cos 6l.$*

If a perpendicular be let fall from the point L in the meridian PQ, upon the semi-axis CP, it will be the radius of the parallel of latitude at that point, and will obviously be equal to the abscissa CM. Hence we have $R_p = R_n \cos l$, the radius of the parallel of latitude, and consequently $D_p = D_n \cos l$, the length of a degree of the parallel at the latitude l. Therefore, multiplying (3) by $\cos l$, we have, for the length of a degree of the parallel at the latitude l in powers of $\cos l$,

(7) $D_p = 111675.89864\cos l - 375.20082\cos^3 l + 1.89079\cos^5 l - 0.01058\cos^7 l.$

But $\cos^3 l = \frac{3}{4}\cos l + \frac{1}{4}\cos 3l$; $\cos^5 l = \frac{5}{8}\cos l + \frac{5}{16}\cos 3l + \frac{1}{16}\cos 5l$; and $\cos^7 l = \frac{35}{64}\cos l + \frac{21}{64}\cos 3l + \frac{7}{64}\cos 5l + \frac{1}{64}\cos 7l$. Substituting these values for the different powers of $\cos l$ in (7), we have, for the length of a degree of the parallel at the latitude l in multiples of $\cos l$,

(8) $D_p = 111399.67397\cos l - 93.21281\cos 3l + 0.11702\cos 5l - 0.00017\cos 7l.$

In practice it will be necessary to retain only three decimal figures in these formulas, and it will be found more convenient to use the powers when we work by the logarithmic, and the multiples when we work by the natural cosines. The following are the formulas for

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* The decimal part of the coefficient of $\cos 2l$ in this formula is wrong in the Report of the Superintendent of the United States Coast Survey, for the year 1853, p. 100.*

the powers and the multiples in French metres, American yards, and English yards.

I. In Powers of cos l.

$$D_m = \begin{cases} 111679.899 - 1125.602\cos^2 l + 9.454\cos^4 l - 0.074\cos^6 l, \text{ in F. metres.} \\ 122129.741 - 1230.925\cos^2 l + 10.339\cos^4 l - 0.081\cos^6 l, \text{ in A. yards.} \\ 122136.829 - 1230.996\cos^2 l + 10.339\cos^4 l - 0.081\cos^6 l, \text{ in E. yards.} \end{cases}$$

$$D_n = \begin{cases} 111679.899 - 375.201\cos^2 l + 1.891\cos^4 l - 0.011\cos^6 l, \text{ in F. metres.} \\ 122129.741 - 410.308\cos^2 l + 2.068\cos^4 l - 0.012\cos^6 l, \text{ in A. yards.} \\ 122136.829 - 410.332\cos^2 l + 2.068\cos^4 l - 0.012\cos^6 l, \text{ in E. yards.} \end{cases}$$

$$D_p = \begin{cases} 111679.899\cos l - 375.201\cos^3 l + 1.891\cos^5 l - 0.011\cos^7 l, \text{ in F. metres.} \\ 122129.741\cos l - 410.308\cos^3 l + 2.068\cos^5 l - 0.012\cos^7 l, \text{ in A. yards.} \\ 122136.829\cos l - 410.332\cos^3 l + 2.068\cos^5 l - 0.012\cos^7 l, \text{ in E. yards.} \end{cases}$$

II. In Multiples of cos l.

$$D_m = \begin{cases} 111120.620 - 558.109\cos 2l + 1.168\cos 4l - 0.002\cos 6l, \text{ in F. metr.} \\ 121518.130 - 610.331\cos 2l + 1.277\cos 4l - 0.002\cos 6l, \text{ in A. yds.} \\ 121525.183 - 610.366\cos 2l + 1.277\cos 4l - 0.002\cos 6l, \text{ in E. yds.} \end{cases}$$

$$D_n = \begin{cases} 111493.004 - 186.660\cos 2l + 0.234\cos 4l, & \text{in French metres.} \\ 121925.359 - 204.126\cos 2l + 0.256\cos 4l, & \text{in American yards.} \\ 121932.434 - 204.137\cos 2l + 0.256\cos 4l, & \text{in English yards.} \end{cases}$$

$$D_p = \begin{cases} 111399.674\cos l - 93.213\cos 3l + 0.117\cos 5l, & \text{in French metres.} \\ 121823.296\cos l - 101.935\cos 3l + 0.128\cos 5l, & \text{in American yards.} \\ 121830.366\cos l - 101.941\cos 3l + 0.128\cos 5l, & \text{in English yards.} \end{cases}$$

ON SPHERICAL ANALYSIS.

By George Eastwood, Saxonville, Mass.

[Continued from Page 304, Vol. II.]

Proposition VI.

To find the intersection of two given great circles of the sphere.

In this proposition it is manifest that the abscissa, X, and the ordinate, Y, of the point of intersection are common to both the given circles. If therefore we define the equation of one of them by

$y = \tau x + \beta$, the equation of the other by $y = \tau' x + \beta'$, and the equation of their common point of intersection by

$$Y = \tau X + \beta = \tau' X + \beta',$$

we shall have $$X = \frac{\beta' - \beta}{\tau - \tau'},$$

and $$Y = \tau X + \beta = \frac{\tau\beta' - \tau\beta}{\tau - \tau'} + \beta = \frac{\tau\beta' - \tau'\beta}{\tau - \tau'},$$

for the point $X\,Y$ of intersection.

PROPOSITION VII.

To find the intersection of a great with a less circle of the sphere.

Let the centre of the less circle be taken for the origin of co-ordinates; and let, as in Cor. 2 of Prop. II., its equation be represented by

$$y^2 + 2xy\cos\omega + x^2 = r^2.$$

If we solve this equation for y, we shall find

$$y = (r^2 - x^2\sin^2\omega)^{\frac{1}{2}} - x\cos\omega.$$

But by (19) $$y = \tau x + \beta,$$

$$\therefore r^2 - x^2\sin^2\omega = (\tau x + x\cos\omega + \beta)^2,$$

$$\therefore r^2 - \beta^2 = (\tau^2 + 2\tau\cos\omega + 1)\,x^2 + 2\beta\,(\tau + \cos\omega)\,x,$$

$$\therefore x = \frac{(r^2(\tau^2 + 2\tau\cos\omega + 1) - \beta^2\sin^2\omega)^{\frac{1}{2}} - \beta(\tau + \cos\omega)}{\tau^2 + 2\tau\cos\omega + 1} = X,$$

$$\therefore y = \tau x + \beta = \frac{\tau(r^2(\tau^2 + 2\tau\cos\omega + 1) - \beta^2\sin^2\omega)^{\frac{1}{2}} + \beta(\tau\cos\omega + 1)}{\tau^2 + 2\tau\cos\omega + 1)} = Y.$$

Now, that this value of y may be possible, it is necessary that we have

$$r^2 > \frac{\beta^2\sin^2\omega}{\tau^2 + 2\tau\cos\omega + 1};$$

and that the two circles may touch each other, or be tangential, it is necessary and sufficient that

(27) $$r = \pm\left(\frac{\beta^2\sin^2\omega}{\tau^2 + 2\tau\cos\omega + 1}\right)^{\frac{1}{2}} = \frac{\pm\beta\sin\omega}{(\tau^2 + 2\tau\cos\omega + 1)^{\frac{1}{2}}}.$$

As r, in this case, is equal to the distance between the pole of the less circle and the circumference of the great circle, it follows that (27) makes known by its tangent the distance from the origin of a great circle whose intercepts on the axes of reference are α and β.

If the axes of reference are rectangular, the condition of contact will become $\beta = \pm r(1+\tau^2)^{\frac{1}{2}}$; so that the equation of a great circle of the sphere tangent to a less circle may be written

$$y = \tau x \pm r(1+\tau^2)^{\frac{1}{2}}. \tag{28}$$

Equation (28) may be applied to the solution of the following problem: —

A less circle of the sphere has its centre at the right angle of a right spherical triangle; it is required to draw a great circle tangent to it, so that the tangents of the intercepted segments of the sides of the triangle may have a given ratio.

There are evidently two solutions to this problem: for if we seek the intersection of the two circles $y = \tau x \pm r(1+\tau^2)^{\frac{1}{2}}$, we see at once that x and y cannot be finite for the two equations

$$y = \tau x + r(1+\tau^2)^{\frac{1}{2}}, \qquad y = \tau x - r(1+\tau^2)^{\frac{1}{2}},$$

when they subsist together. That they may subsist, we must obviously have $x = \infty$; that is, the intersection must be 90° from the pole of the less circle. Hence, in this case, we must have recourse to the latitude of the point of intersection.

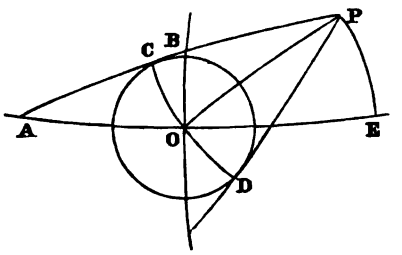
Take $\bar{y}$ for the latitude of the point of intersection, then by (5) $y = \frac{\bar{y}}{\cos x}$; so that the equations of the two tangent circles may be written

$$\bar{y} = \tau \sin x + r(1+\tau^2)^{\frac{1}{2}} \cos x, \qquad \bar{y} = \tau \sin x - r(1+\tau^2)^{\frac{1}{2}} \cos x,$$

from which we readily deduce $\cos x = 0$, or $x = \frac{\pi}{2}$, and $\therefore \bar{y} = \tau$.

As a further illustration of the subject, let P, in the annexed

diagram, be the point of intersection of the two tangent circles, C and D their points of contact with the given circle, and AO, OB the intercepted segments of the sides of the given triangle. Then, because $OP = OE$ is a quadrantal arc of a great circle, and the angle



$$POC = \angle POD = \frac{\pi}{2},$$

we have the latitude

$$PE = \angle POE = \frac{\pi}{2} - \angle DOE = \frac{\pi}{2} - \angle AOC.$$

And, because the angles AOB, ACO, BCO are each right angles, the spherical right triangles ACO, BCO give

$$\tan CO = \tan AO \cdot \cos AOC = \tan BO \cdot \cos BOC = \tan BO \cdot \sin AOC;$$

$$\therefore \cot AOC = \frac{\tan BO}{\tan AO} = \frac{\beta}{-\alpha} = \tau = \tan PE.$$

If $x'y'$ be the co-ordinates of the point of contact C or D, the equation of the tangent AP or PD may be written

$$yy' + xx' = r^2. \tag{29}$$

For, by (28), the equation of the point of contact is

$$y' = \tau x' + r^2 (1 + \tau^2)^{\frac{1}{2}},$$

in which $r = (x^2 + y^2)^{\frac{1}{2}}$.

Resolving therefore for τ, we find

$$\tau = -\frac{x'}{y'} = \frac{-1}{\frac{y'}{x'}} = -\frac{\beta}{\alpha},$$

the substitution of which in (28) gives $yy' + xx' = r^2$.

PROPOSITION VIII.

To find the distance between two given points on the surface of the sphere.

Designating the latitudes of the given points by $\bar{y}_1, \bar{y}_2$, the longi-

tudes by x', x'', and their distance apart by Δ, we shall have from the fundamental spherical triangle,

$$\cos \Delta = \sin \bar{y}_1 \sin \bar{y}_2 + \cos \bar{y}_1 \cos \bar{y}_2 \cos (x' - x''),$$
$$= \frac{\bar{y}_1 \bar{y}_2 + \cos x' \cos x'' + \sin x' \sin x''}{(1+\bar{y}_1^2)^{\frac{1}{2}} (1+\bar{y}_2^2)^{\frac{1}{2}}};$$

in which $\sin \bar{y}_1 \sin \bar{y}_2$ and $\cos \bar{y}_1 \cos \bar{y}_2$ are replaced by their conventional equivalents $\bar{y}_1 \bar{y}_2$ and $(1+\bar{y}_1^2)^{\frac{1}{2}} (1+\bar{y}_2^2)^{\frac{1}{2}}$.

If now we introduce, for the geographical co-ordinates $\bar{y}_1$ and $\bar{y}_2$, the geometrical co-ordinates $y' \cos x'$ and $y'' \cos x''$, or their equals, $\frac{y'}{(1-x'^2)^{\frac{1}{2}}}$ and $\frac{y''}{(1-x''^2)^{\frac{1}{2}}}$, we shall have

$$(30) \qquad \cos \Delta = \frac{y' y'' + x' x'' + 1}{\pm (1+x'^2+y'^2)^{\frac{1}{2}} (1+x''^2+y''^2)^{\frac{1}{2}}},$$

$$(31) \qquad \sin \Delta = \frac{\{(x''-x')^2 + (y''-y')^2 + (x' y'' - x'' y')^2\}^{\frac{1}{2}}}{(1+x'^2+y'^2)^{\frac{1}{2}} (1-x''^2+y''^2)^{\frac{1}{2}}},$$

$$(32) \qquad \tan \Delta = \frac{\pm \{(x''-x')^2 + (y''-y')^2 + (x' y'' - x'' y')^2\}^{\frac{1}{2}}}{y' y'' + x' x'' 1}.$$

In these formulas the double signs indicate the distance of the point $x' y'$ from the point diametrically opposite to the point $x'' y''$, as well as from the point $x'' y''$ itself. This will be obvious when it is remembered that two points on the surface of the sphere may be joined by two different arcs of great circles, which, taken together, make up the whole circumference of a great circle.

If in equation (32) we regard Δ as constant, and x', y', as variables, we shall have the equation of a less circle of the sphere referred to an external origin, in functions of the co-ordinates x'', y'', of its pole, and of its radius Δ.

When two points on the surface of the sphere are at a distance of 90° from each other, the cosine of this distance is zero; we shall therefore have, between their co-ordinates x', y', x'', y'', the relation

$$(33) \qquad y' y'' + x' x'' + 1 = 0.$$

When two great circles intersect each other at right angles, their poles will be 90° apart; hence the relation (33) will hold for the co-ordinates of their poles.

If two great circles are represented by the equations

$$Mx + Ny = P, \qquad M'x + N'y = P',$$

the relation (33) will give

$$MM' + NN' + PP' = 0. \tag{34}$$

And, generally, if two great circles be defined by the equations

$$\frac{y}{\beta} + \frac{x}{\alpha} = 1, \qquad \frac{y}{\beta'} + \frac{x}{\alpha'} = 1;$$

the relation which expresses that they cut each other at right angles will be truly defined by

$$\frac{1}{\beta'\beta} + \frac{1}{\alpha'\alpha} + 1 = 0. \tag{35}$$

PROPOSITION IX.

To find an expression for the angle formed by the intersection of two great circles of the sphere.

The angle formed by the intersection of the two circles is evidently equal to the distance between their poles; ∴ if θ represents this distance, and if $x', y', x'' y''$, be the co-ordinates of the poles, equation (30) will give

$$\cos\theta = \frac{y'y'' + x'x'' + 1}{(1 + x'^2 + y'^2)^{\frac{1}{2}}(1 + x''^2 + y''^2)^{\frac{1}{2}}},$$

for the cosine of the required angle.

But if the circles be represented by $\alpha y + \beta x = \alpha\beta$ and $\alpha' y + \beta' x = \alpha'\beta'$, Prop. IV., gives

$$x' = -\frac{1}{\alpha}, \quad y' = -\frac{1}{\beta} \quad \text{and} \quad x'' = -\frac{1}{\alpha'}, \quad y'' = -\frac{1}{\beta'}.$$

Hence

$$\cos\theta = \frac{\alpha\alpha' + \beta\beta' + \alpha\alpha'\beta\beta'}{(\alpha^2 + \beta^2 + \alpha^2\beta^2)^{\frac{1}{2}}(\alpha'^2 + \beta'^2 + \alpha'^2\beta'^2)^{\frac{1}{2}}}. \tag{36}$$

Equation (36) shows that if

$$\alpha\alpha' + \beta\beta' + \alpha\alpha'\beta\beta' = 0, \tag{37}$$

the two circles will be perpendicular to each other.

PROPOSITION X.

To find the distance of a given point on the surface of the sphere from the circumference of a great circle.

Let the given point be designated by $x'y'$, the pole of the given circle by $x''y''$, the distance between them by Δ, and the required distance by Δ'; then, by (32),

$$\tan\Delta = \frac{\pm\{(x''-x')^2+(y''-y')^2+(x'y''-x''y')^2\}^{\frac{1}{2}}}{y'y''+x'x''+1}.$$

But $\tan\Delta' = \cot\Delta$, hence

$$\tan\Delta' = \frac{y'y''+x'x''-1}{\pm\{(x''-x')^2+(y''-y')^2+(x'y''-x''y')^2\}^{\frac{1}{2}}}, \tag{38}$$

makes known the required distance.

If for x'' and y'' we substitute their values, $-\frac{1}{\alpha'}$ and $-\frac{1}{\beta'}$, then (38) becomes

$$\tan\Delta' = \frac{-\alpha'y'-\beta'x'+\alpha'\beta'}{\pm\{(\alpha'x'-1)^2+(\beta'y'-1)^2+(\alpha'x'-\beta'y')^2\}^{\frac{1}{2}}}, \tag{39}$$

Again, since

$$\sin\Delta = \frac{\tan\Delta'}{(1+\tan^2\Delta')^{\frac{1}{2}}} = \frac{-\alpha'y'-\beta'x'+\alpha'\beta'}{\pm\{(\alpha'^2+\beta'^2+\alpha'^2\beta'^2)(1+x'^2+y'^2)\}^{\frac{1}{2}}},$$

$\therefore$ in general, if the equation of a great circle be represented by $\mu y + \nu x + \varrho = 0$, then a perpendicular upon its circumference from a given point, $x'y'$, may be determined from the equation

$$\sin\Delta = \frac{\mu y+\nu x+\varrho}{\mp\{(\mu^2+\nu^2+\varrho^2)(1+x'^2+y'^2)\}^{\frac{1}{2}}}. \tag{40}$$

PROPOSITION XI.

To find the equation of a great circle passing through a given point, and perpendicular to another given great circle.

Let the given great circle be defined by the equation

$$(a) \qquad \frac{y'}{\beta'} + \frac{x'}{\alpha'} - 1 = 0,$$

and the required one by the equation

$$(b) \qquad \frac{y}{\beta} + \frac{x}{\alpha} - 1 = 0.$$

Then, that the conditions of the problem may be fulfilled, we must have, by (35) and (23),

$$(c) \quad \frac{1}{\alpha\alpha'} + \frac{1}{\beta\beta'} + 1 = 0, \quad y - y' = \tau'(x - x') = -\frac{\beta'}{\alpha'}(x - x').$$

Now, from (b) we find

$$(d) \qquad \alpha = \frac{\beta x}{\beta - y},$$

and from (a) we find $\quad \alpha' = \frac{\beta' x'}{\beta' - y'};$

$$(e) \qquad \therefore \alpha\alpha' = \frac{\beta\beta' x x'}{(\beta - y)(\beta' - y')}.$$

Hence (c) becomes $\quad \frac{(\beta - y)(\beta' - y')}{\beta\beta' x x'} + \frac{1}{\beta\beta'} + 1 = 0,$

which gives $\quad \beta' = \frac{(\beta - y) y' - x' x}{\beta - y + \beta x' x} = \frac{\beta y' - \alpha x'}{(1 + \alpha x')\beta},$

by virtue of (d).

Again, from (e) we have

$$\alpha' = \frac{\beta\beta' x x'}{(\beta - y)(\beta' - y')\alpha} = \frac{\{(\beta - y) y' - x' x\}\beta}{-(\beta - y)(1 + \beta y')\alpha} = \frac{\beta y' - \alpha x'}{-(1 + \beta y')\alpha}.$$

Hence $\tau' = -\frac{\beta'}{\alpha'} = \frac{\alpha(1 + \beta y')}{\beta(1 + \alpha x')}$, and therefore the required equation is

$$(41) \qquad y - y' = \frac{\alpha(1 + \beta y')}{\beta(1 + \alpha x')}(x - x').$$

If the co-ordinates of the pole of the given circle be x'' and y'',

then, since $\alpha = -\frac{1}{x''}$ and $\beta = -\frac{1}{y''}$, equation (41) gives

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$

which accords with (25), as it evidently ought to do.

A TREATISE ON DETERMINANTS.

CHAPTER I.

INTRODUCTORY.

1. If, by the usual methods of elimination, we solve the linear equations $a'x + b'y = m'$, $a''x + b''y = m''$, we get

$$x = \frac{m'b'' - m''b'}{a'b'' - a''b'}, \qquad y = \frac{a'm'' - a''m'}{a'b'' - a''b'};$$

whose numerators and common denominator have each that peculiar "alternate symmetry," whereby either changes merely in sign if we transpose any two letters throughout without moving the indices; e. g. $(m'b'' - m''b'') = -(b'm'' - b''m')$. So also, if we transpose two indices throughout without moving the letters; e. g. $(m'b'' - m''b') = -(m''b' - m'b'')$. Indeed the symmetry of either polynomial is precisely the same with respect to its indices as with respect to its letters. Moreover, the numerator of x or of y is formed from the common denominator, by writing m instead of a or instead of b.

All these properties belong as well to any n values x, y, &c., determined by n given linear equations. Thus,

$$x = \frac{m'b''c''' + m''b'''c' + m'''b'c'' - m'''b''c' - m''b'c''' - m'b'''c''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''},$$

$$y = \frac{a'm''c''' + a''m'''c' + a'''m'c'' - a'''m''c' - a''m'c''' - a'm'''c''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''},$$

$$z = \frac{a'b''m''' + a''b'''m' + a'''b'm'' - a'''b''m' - a''b'm''' - a'b'''m''}{a'b''c''' + a''b'''c' + a'''b'c'' - a'''b''c' - a''b'c''' - a'b'''c''},$$

which evidently satisfy the equations

$$a'x+b'y+c'z=m', \quad a''x+b''y+c''z=m'', \quad a'''x+b'''y+c'''z=m''',$$

have these properties.

Again, the numerators and denominator of the set of values obtained from n equations may be found, by a simple law, from the values that satisfy sets of $(n-1)$ equations. Thus the above common denominator of x, y, z, may be written

$$a'(b''c'''-b'''c'')+a''(b'''c'-b'c''')+a'''(b'c''-b''c'),$$

or

$$a'(b''c'''-c''b''')+b'(c''a'''-a''c''')+c'(a''b'''-b''a''');$$

whose parentheses are the denominators of those values that satisfy the equations

$$b'\xi+c'\eta=m', \quad b''\xi+c''\eta=m'', \quad b'''\xi+c'''\eta=m''';$$

or else the equations

$$a''\xi+a'''\eta=A, \quad b''\xi+b'''\eta=B, \quad c''\xi+c'''\eta=C,$$

taken two and two.

When the common denominator vanishes, some or all of $x, y, \ldots$ must become infinite, indicating that the given equations are incompatible; unless all $m', m'', \ldots$ likewise vanish, in which case $x, y, \ldots$ are indeterminate, indicating that some of the given equations are deducible from the rest.

2. These numerators and denominators are called DETERMINANTS. Their symmetry should appear in most investigations where systems of linear equations enter; and their vanishing commonly indicates that certain co-efficients or functions are not independent. Their properties will often aid us when the difficult or characteristic feature of a problem is to eliminate among the equations or to combine the results of elimination.

The subject of Derivatives assumes a linear character, [we may

say, through the systematic omission of the higher infinitesimals,] such that the change of systems of independent variables in differentiating or integrating, the assignment of multiple definite integrals, and the properties of linear differential equations, so important in dynamics, involve determinants.

In Algebra, equations of any degree have been treated as linear equations, whose new variables are powers and products of the old ones, and may be eliminated by determinants. By such elimination among equations derived from a single one by differentiation, we obtain determinants; e. g., the "discriminant" whose vanishing shall express certain equalities among the roots, and which are "invariants." The extensive theory of invariants, &c., has also other connections with determinants.

In Geometry, we naturally work by combining right lines and planes, whose equations are linear. Co-ordinates are often changed by "linear substitution," as in Perspective, and this introduces Invariants, &c. The contents of triangles and pyramids, being definite integrals, are expressed by determinants.

In the Theory of Numbers and elsewhere, it is natural that those determinants whose vanishing changes the character of any problem, should likewise often affect it by their form or smallness when they do *not* vanish. Especially, the discriminant of a quadratic form so *determines* the character of that form, as to have first received from GAUSS the name "determinant." (Disq. Arith. § ...)

The preference of linear forms, even in Geometry and by Nature in Dynamics, appears to be not merely because linear equations are the simpler, as quadratics are simpler than cubics; but also because, unlike all higher equations, they give but one value to each variable, which is always real if the co-efficients and absolute terms are so, the degree, too, being never raised by eliminating any of the variables.

3. Though regarded chiefly as a rather simple instrument of investigation, Determinants have also an interest of their own. I shall present what I suppose to be their leading features and their general modes of application.*

What follows will be chiefly from other treatises; most from BRIOSCHI, but much also from SALMON, PEIRCE, FAÀ DE BRUNO, SPOTTISWOODE, and from several memoirs of CAYLEY and others; with occasional suggestions of my own. Writing rather of the theory than of its authors, I could not always mention the latter without more reading, neither do I care to keep their work unchanged.

I shall try to state clearly, in coarse type, what is at first most important; giving more concisely, and in finer print, things which the reader may omit at his option, as the fine print will never be needed to demonstrate anything that follows in coarse. But he would best notice any *examples* in small type which follow and illustrate the large.

Whether for mere *reference*, or to indicate final *results*, formulas, and perhaps verbal statements, will be numbered in one series, thus, (1), (2), ... Some important results may be marked by bold figures, thus, (**2**). Analogous statements may be indicated by accents, as, (2), (2′), (2″). Subscript indices show which members of an equation are referred to; e. g., (2_3) denotes the third member of (2), and (2_{135}) denotes the equality $(2_1) = (2_3) = (2_5)$.

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* For further and instructive examples, see COMBESCURE'S French translation of the admirable "Theory of Determinants and their principal Applications," by Prof. BRIOSCHI, of Pavia, 1856, 8vo, pp. 216; for the best elementary exposition of Invariants, &c., see SALMON'S Higher Algebra, Dublin, 1859, 8vo, pp. 147; for a condensed and able account of Determinants as applied to Differential Equations, see PEIRCE'S Analytic Mechanics, 1855, Chapter X.; † and for the "General Theory of Elimination," see FAÀ DE BRUNO'S work, Paris, 1859, pp. 224; SPOTTISWOODE'S *Elementary Theorems relating to Determinants*, 1851, 4to, pp. 63.

† Almost the only other American notice of Determinants which I have seen, is an ingenious and suggestive article "On the Resolution of Symmetrical Equations," by Mr. STOCKWELL, *Astronomical Journal*, VI., 145, 1860.

I have given much thought to the determinant-notation, and especially to that of Chapter III. I have sought to select it from existing systems; but, as the best of these do not perfectly harmonize, I have ventured upon some changes, which, if found conducive to symmetry and convenience, will, I hope, be adopted by others.

Notation should be clearly separated into "temporary," adopted but for a single essay or part of one, and requiring redefinition if ever used again; and "permanent," to be used in the same sense in all memoirs by all writers who adopt it, even without redefinition,— yet always changeable *temporarily and by express definition.* But one notation may be permanently used in different senses, if they differ so *widely* or pertain to such different *subjects* as not to interfere. Of course, I have been most careful with such *permanent* notation as I *have not taken from others.*

Further explanations may be given in text or foot-notes where they are needed. Let us now commence our subject again.

[To be Continued.]

Mathematical Monthly Notices.

Concise Mathematical Operations; being a Sequel to the Author's Class-Books. With much additional Matter. By H. N. ROBINSON, LL.D., formerly Professor of Mathematics in the United States Navy, &c. New York: Ivison and Phinney. 1854.

Elements of Analytical Geometry, and the Differential and Integral Calculus. By H. N. ROBINSON, LL.D., Author of a Course of Mathematics, &c. New York: Ivison and Phinney, 48 and 50 Walker Street.

We include these two volumes in the same notice, because in the former, the author first presented his peculiar views of the Differential Calculus, about which we are at present mainly concerned. The latter work, as we learn from the publisher's advertisement, is now undergoing revision, and the two subjects which it contains will be published in separate volumes in January, 1861. As this will give the author a good opportunity to make such changes and improvements as the use of the work may have suggested, we beg to call his attention to a few features of the present edition, feeling desirous of doing all in our power to improve the text-

books from which our students get their first ideas of the science. In the Preface to the work on Mathematical Operations, the author says: —

"There has been a great deal of unnecessary controversy about the Differential and Integral Calculus, which we think can and ought to be wiped away. And we have here given a little foretaste of what we shall attempt if circumstances prompt us to write a work on that subject.

"It is not for us to assume that we can make science clearer than others, but we have yet to see the works of an author who has made the least attempt to show the simple elementary nature of this science. They at once commence with the definition of constants and variables, and then direct what to do.

"We have yet to see the first book that expends a word in giving an idea of what the Calculus is, or what is the utility and object of the science, and we charge more than half the obscurity to this fact alone: hence we could not forbear being a little elementary when we came to that subject, and we leave it to those readers, who have formerly studied other works on this science, to say whether we have or can dispel any of the obscurity that has so long hovered around it."

Again, on page 259: —

"One reason for the appearance of this work is that it is required, because able mathematicians have written so obscurely. They seem to have written as I should, were I indifferent whether the reader, or rather the learner, understood me or not."

We propose now to allow the author to state his own case; and then we can judge whether the "unnecessary controversy" has been "wiped away," either in the "little foretaste" where "we could not forbear being a little elementary," or in the work, which "circumstances" prompted.

The assurance with which the subject is approached, and the estimate which is put upon the labors of those who founded the science, would naturally lead the reader of the above passages to anticipate either great success or great failure, according to his knowledge of the subject and his acquaintance with the labors of the great masters of the science. We shall quote the exact language, italics, and punctuation of the author; and first from the former work: —

"The *differential calculus* is the science of minute variations, or of corresponding *small differences* — a science which owes its birth to the varying elements of Astronomy." — p. 302.

One of the author's correspondents asked him why he called a certain explanation an application of the differential calculus, to which he replied as follows: —

"If the operation I sent you is not calculation, I know not what it is — it may therefore be called calculus; and if in any operation small quantities may be omitted on account of their insignificance in relation to larger quantities, the small difference so omitted constitutes the *differential calculus*, and to obtain that general expression, you will see, by looking on page 193 of the Astronomy, that the powers of r above the first were omitted." — p. 314.

Arithmetic may therefore be called calculus; and because we omit decimal quantities, which on account of their smallness do not affect the required accuracy of the result, it is the differential calculus.

"The differential calculus is a branch of Analytical Geometry. It is a science for computing the *ratio* of small differences." — p. 315.

After giving the differentials of the circular functions, the author says: —

"All can understand how these equations are obtained; but what if we can? says the inquiring student. What use are they? What do we learn by them? It is useless to answer these questions by words only, we must show the answer by the following examples. Because the differential of a logarithm is the differential of the quantity divided by the quantity, therefore $d \log \sin x = \cot x\, dx$. For example, if we assume $x = 25°$, and also assume $dx = 1'$, the differential, or difference between the log sin 25° and the log sin 25° 1′ is expressed by $m \cot 25° \times 1'$. We might assume $dx = 2'$ as well as $1'$, without error as far as six places of decimals; but it would not do to assume $dx =$ any large number of minutes; hence the differential calculus must be applied with judgment. Those who are naturally more nice than wise are commonly prejudiced against this science, and such frequently say it is no science at all; however, their objections are of no consequence." — pp. 322, 323.

Is it possible that the author does not know the use of TAYLOR'S Theorem?

We shall now give a few extracts from the *Elements of the Calculus*. As a first example the

author takes the equation $y = ax + b$, and, calling h the increment of x, finds $y' - y = a\,h$. He then says: —

"The first member of this equation is obviously the *increment* of y, whatever be the value of h, and when h is *extremely* small in relation to x, (we will not say *infinitely* small, as that word puzzles, and it is unnecessary) then $y' - y$ is extremely small in relation to y and in that case we write dy in place of $y' - y$, and dx in the place of h." — p. 146.

Here, then, we have the author's theory of the Calculus. About how small he supposes a differential to be we are not informed; at any rate it is unnecessary, it seems, so far as the logic of the subject is concerned, to consider them infinitesimals. Perhaps we are expected to use our judgment here as well as in the applications.

"The terms *fluxion*, *differential*, and *derivative*, all mean the same thing. There are cases, as in *derived polynomials* in algebra, that it would not do to call the derivative a differential, as the increment might be too large." — p. 146.

We might suppose, were it not for the unfortunate reference to derived polynomials in algebra, that this supposed equality of meaning assigned to the derivative was a slip of the pen. But, at any rate, we learn from this quotation that an increment may be "too large."

On pages 149 and 150 we find a reference to the *limiting ratio*, but then it is said to be between the function and its variable. But this simple reference to another theory seems little more than accidental; for when we come to the differentiation of fractions on page 155, the author has got back upon the doctrine of "extremely small (not to say infinitely small)."

Such is our author's theory of the differential calculus; which seems to us strikingly like the calculus of finite differences used as the differential calculus. Indeed, if we were asked, as we have been, to give a definition of our author's differential calculus, we should say that it is a method of finding a change in a function corresponding to a given change in the variable, provided the change in the variable be not too great; in other words, it is a rather coarse way of computing changes in the values of functions using only first differences, and the judgment as to how large we may assume those differences to be.

Next, let us examine some specimens of demonstration, in the light of the theory: —

"Sometimes we have $u = f(y)$, and $y = F(x)$, and require the differential coefficient between the function u and the variable x. For that purpose we first find $\frac{du}{dy}$ from one equation, and then $\frac{dy}{dx}$ from the other, and multiply them together, and we have $\frac{du}{dx}$ as required." — p. 154.

It seems, then, that $\frac{du}{dy}$ and $\frac{dy}{dx}$ are the same as two ordinary fractions, and their product must be reduced to its lowest terms to obtain the required result.

How it follows, that in general, dy, as an arbitrary increment, equals dy as a dependent increment, and especially so, if neither is infinitely small, we are not informed. Indeed it is not a self-evident proposition applicable to functions involving any number of variables, in any theory of the calculus; and the difficulty is simply disguised for the sake of an apparent simplicity, which will deceive none but very young students.

"Hence we observe that whatever value is given to the variable y, *Taylor's theorem*, and the *binomial theorem*, will give the same result. Hence, when one of these theorems *fail* the other *fails*, but we never apply the word fail to the binomial theorem, and it is not clear to us that such an expression should ever be applied to Taylor's theorem."

If it be not clear to the author, how will it be to the student? The remark that "the *failure* is in the hypothesis and not in the theorems," shows that the author was probably correct when he said, "it is not clear to us."

The true question is, Does TAYLOR'S Theorem, for any function whatever, fail to give the true development to any number of terms for a given hypothesis, when some other method would not fail?

On pages 199 – 202 the tangent, sub-tangent, &c. of plane curves are found by means of the Differential Calculus, and on page 202 we find the remark that, "the student will perceive that these results are here obtained far more easily than in analytical geometry, but we are indebted to analytical geometry for the primary equation of the curve." But a note on the bottom of the same page says that, "It is important that the student should observe that this portion of the calculus is pure analytical geometry."

The idea that the calculus is a branch of analytical geometry is also impressed upon the student on pages 209, 217, 220, 229, as well as in the Preface, apparently, without the faintest suspicion that, as a calculus of functions, with its fundamental principles and its logic, it is entirely independent of any of its applications.

"The differential of an increasing quantity is *positive* before arriving at the maximum *zero*, and *negative* afterwards as we perceive by merely inspecting the figure, and *this is a general principle*. In like manner the differential of a decreasing quantity is minus before it attains its minimum point, it is zero at that point, and positive after passing that point. Hence, if the second differential of a function is *minus*, it indicates that the first differential corresponds to a *maximum*, and if *plus* it indicates a *minimum*. The foregoing illustrations are too plain and practical to meet the entire approbation of some minds; therefore, we give the following as more general and abstract."

Why "some minds" should object to correct geometrical illustrations, and what kind of minds they must be if they do, we cannot understand; but we can comprehend why the "more general and abstract" discussion should not meet the entire approbation of some minds; for a cruder one we never examined. Some of the important points are entirely omitted; such, for instance, as the case in which the same values of the variable reduce the second derivative, as well as the first, to zero. Nothing is said of the maxima and minima of functions of more than one variable.

But we must, at present, omit many other points in the volumes before us for want of space. We have given them, and especially the parts devoted to the Differential and Integral Calculus, a careful and thorough examination; and we are constrained to say that we have never had the misfortune to examine two works, so totally unfitted for text-books, both as regards the treatment of the subjects and the style in which they are written.

We shall take an early opportunity of calling attention to other works by the same author.

Objects and Plan of an Institute of Technology; including, I. A Society of Arts; II. A Museum of Arts; and III. A School of Industrial Science. Proposed to be established in Boston. Prepared by direction of the Committee of Associated Institutions of Science and Arts; and addressed to Manufacturers, Merchants, Mechanics, Agriculturists, and other friends of enlightened industry in the Commonwealth.

WILLIAM B. ROGERS, *Chairman.*

MARSHALL P. WILDER,	JAMES M. BEEBE,	B. S. ROTCH,
SAMUEL H. GOOKIN,	DR. S. CABOT, Jr.,	J. D. PHILBRICK,
ALFRED ORDWAY,	G. W. PRATT,	GEORGE B. EMERSON,
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ALEX. H. RICE,	DR. S. KNEELAND, Jr.,	ERASTUS B. BIGELOW,
E. S. TOBEY,	CHARLES L. FLINT,	CHARLES H. DALTON,

Committee.

The Plan contemplates the following Standing Committees. *First.* Committees of Arts. On Mineral Materials; on Organic Materials; on Tools and Instruments; on Machinery and Motive Powers; on Texile Manufactures; on Manufactures of Wood, Leather, Paper, India Rubber, &c.; on Manufactures of Pottery, Glass, and Precious Metals; on Chemical Products and Processes; on Household Economy, including Warming, Illumination, Water-Supply, Ven-

tilation, and the Preparation and Preservation of Food; on Engineering and Architecture; on Commerce, Navigation, and Inland Transport; on the Graphic and Fine Arts. *Second.* Committee on the Museum. *Third.* Committee on the School of Industrial Science. *Fourth.* Committee on Publications. A very well defined outline of the duties of each committee is given; subject to such changes as experience shall hereafter suggest.

The School of Industrial Science and Art, is to be arranged under the following departments: 1st. A School of Design; 2d. A School of Mathematics; 3d. A School of Physics; 4th. A School of Chemistry; 5th. A School of Geology. Such is the general outline of a most admirable plan, with the details so arranged as to form a symmetrical and systematic whole.

The Report begins by setting forth in a clear and forcible manner the need now felt for the establishment of such an Institution, and the practical value of the results which would flow from it. In the following opinion we heartily concur: "In New England, and especially in our own Commonwealth, the time has arrived when, as we believe, the interests of Commerce and the Arts, as well as of General Education, call for the most earnest co-operation of intelligent culture with industrial pursuits. Our success hitherto in the competitions of trade, manufactures, and the other productive Arts, has been the admitted result of the superior intelligence which has inspired our enterprise and guided our activity; but, to secure a steady prosperity in the midst of the busy inventions and rapidly expanding knowledge which mark these pursuits in the leading European nations, we feel that it has become indispensable for us to provide, at least as effectually as they have done, such facilities for practical knowledge, and for the intelligent guidance of enterprise and labor, as may make our progress commensurate, step by step, with the advances of scientific and practical discovery."

We regret that space forbids our giving a full statement of the arrangement of the proposed Museum, and School of Industrial Science and Art. But the whole plan is well matured, and eminently practical; and of the desirableness of establishing such an Institution in Boston, there can be no question. Still, admitting all that its most enthusiastic friends claim for it, can the idea be carried into execution? We believe it can and will be done. It will need wise counsellors, and unyielding friends, who will be satisfied with nothing less than the largest measure of success. But these it has the good fortune to possess in the persons of the Committee under whose direction this comprehensive Report has been prepared. It will also need the most ample means; but these, we are assured by those who know, and have a right to speak in its behalf, will not be wanting; and it only remains to be seen whether our State will add this last and crowning excellence to her unsurpassed educational system by granting the desired location, upon which private munificence will rear the Institution.

On a Chart and Diagram for Facilitating Great-Circle Sailing. By HUGH GODFRAY, M. A., St. John's College, Cambridge. (From the Transactions of the Cambridge Philosophical Society, Vol. X. Part II.)

On charts, constructed on the *Central or Gnomonic Projection*, great circles are represented by straight lines; and thus, in Great-Circle Sailing, such charts answer the same purpose that Mercator's Charts do in Rhumb Sailing. To see the ship's track it is only necessary to draw a straight line from the ship's place to her destination. The author says: "By the addition of a Course and Distance Diagram I have made this projection answer all the conditions of Great-Circle Sailing with as much, if not more, facility than Mercator's chart does for sailing on a Rhumb. The track is *seen* a straight line, and this being drawn, the various courses and the distances to be run upon each are obtained, as also the distance from the ship to her destination, by a mere inspection of the diagram; and the chart can be used like an ordinary one for pricking off the ship's place from day to day."

Accompanying the Chart and Course and Distance Diagram, we have received from the

author a *Time Azimuth Diagram*, with a short memoir in explanation of its object, use, and construction. By means of the Diagram the true azimuth can be obtained without calculation to within one eighth of a degree. The data required are the latitude, sun's declination and apparent time at ship. It is hardly possible that such simple and elegant graphic solutions of these problems should not soon find their way into general use.

On the Secular Variations and Mutual Relations of the Orbits of the Asteroids. By SIMON NEWCOMB, S. B., A.A.S. (From the Memoirs of the American Academy, New Series, Vol. VIII.) Cambridge: Welch, Bigelow, and Co., Printers to the University. 1860.

The particular object of this Memoir is to submit OLBER'S hypothesis of the origin of the asteroids to as rigorous a test as is consistent with the nature of the problem. If this hypothesis be true, then there must have been, and still be, certain relations among the asteroid orbits; and the test consists in determining whether these relations do in fact exist. The author has arranged the discussion under the following heads: —

§ 1. *Computation of the rigorous expressions in terms of the times of the elements of the asteroids.*

This section is treated with great elegance and skill. All the terms in these expressions, depending upon the elements of the asteroids, are given in tables with the mean distance of the asteroid as an argument; so that other asteroids may, with but trifling labor, be added to the list of twenty-five treated in this memoir, as fast as their elements become sufficiently well known.

§ 2. *Of the possibility that the orbits of all the asteroids once intersected in a common point.*

Carrying back the elements of the asteroid orbits by means of the formulas in § 1, there is found no tendency to a common point of intersection of the orbits.

§ 3. *Have the elements of the asteroid orbits ever been materially affected by a resisting medium?*

Not sensibly.

§ 4. *Of the relations among the mean distances, eccentricities, and inclinations of the orbits of the asteroids; and between their masses and the velocities with which they must have been projected, if* OLBER'S *hypothesis be true.*

The actual relations are not found to be those which would exist if OLBER'S hypothesis be true.

§ 5. *Of certain observed relations among the asteroids which are the necessary or probable result of known causes, and therefore throw no light on the origin of the asteroids.*

This last section is very interesting, and contains satisfactory solutions of several important questions; such as the unequal distribution of the nodes and perihelia of the orbits; the interlinking of certain orbits, &c.

This valuable memoir is by far the most successful discussion ever made to settle one of the most deeply interesting problems in astronomy, which has agitated the scientific world for the last half-century; and will reflect great credit upon the ability of its author.

Plane Trigonometry, crown 8vo, pp. 271; *Spherical Trigonometry*, crown 8vo, pp. 112. For the use of Colleges and Schools. With numerous examples. By I. TODHUNTER, M. A., Fellow and Assistant Tutor of St. John's College, Cambridge. Macmillan and Co.: Cambridge, and 23 Henrietta Street, Covent Garden, London. 1859.

We do not call the attention of teachers and students to these volumes because we think that there are no good text-books upon these subjects in this country. On the contrary, we have a few excellent ones; and we have no hesitation in saying that we know of no foreign work which surpasses PROFESSOR CHAUVENET'S *Treatise on Plane and Spherical Trigonometry* in all the elements of a good text-book, except, perhaps, in its collection of examples for practice. In this respect these volumes are especially worthy the attention of teachers; the one on Plane Trigonometry contains over six hundred examples, and the other about one hundred and fifty. But they are not ordinary works in other respects. We are particularly interested in the

chapters on the Construction of Trigonometrical Tables; Theory of Proportional Parts; Expansions of Trigonometrical Functions; Summation of Trigonometrical Series; Resolution of Trigonometrical Expressions into Factors; Area of a Spherical Triangle. Spherical Excess; Geodetical Operations.

The author, after showing the connection of the values of the old and new Trigonometric Functions, remarks that, "The old definitions give some indications of the origin of the terms *sine*, *cosine*, &c." The word *sine* seems derived from the Latin *sinus*, a bosom, the arc is supposed to represent a bow, and thus gets its name, and the string, half of which represents the sine of half the arc, would come against the breast of the archer. The words *tangent* and *secant* are naturally derived from the old definitions. (See *Penny Cyclopædia*, Art. *Trig.*)

The modern method has now completely superseded the ancient method in English works; it was introduced by DR. PEACOCK. (See PEACOCK'S *Algebra*, Vol. II. p. 151.) It may, however, be observed, that it is stated by PROFESSOR DE MORGAN (*Trig.* and *Double Algebra*, p. 18), that RHETICUS, who gave the first complete trigonometrical table, and invented the secant and cosecant to complete it, used the method of ratios."

We find in the *Spherical Trigonometry*, p. 38, the method of proof of NAPIER'S Rules indicated by NAPIER in his *Mirifici Logarithmorum Canonis Descriptio*, pp. 32, 35.

In this connection the author gives a collection of opinions with respect to the practical utility of NAPIER'S Rules. "In the whole compass of mathematical science there cannot be found, perhaps, rules which more completely attain that which is the proper object of rules, namely, facility and brevity of computation." WOODHOUSE'S *Trig.*, Chap. X. "In the opinion of DELAMBRE (and no one was better qualified by experience to give an opinion) these theorems are best recollected by the practical calculator in their unconnected form." (AIRY'S *Trig.*, *Encyclopædia Metropolitana.*) "There are certain mnemonical formulæ called NAPIER'S *Rules of Circular Parts*, which are generally explained. We do not give them, because we are convinced that they only create confusion instead of assisting the memory."

We had always supposed by dividing a circle into five parts, and writing upon them in order, a, $\frac{\pi}{2} - B$, $\frac{\pi}{2} - c$, $\frac{\pi}{2} - A$, b; then selecting any one of these five quantities, and observing that of the remaining four, two (b, $\frac{\pi}{2} - B$) are adjacent and two ($\frac{\pi}{2} - c$, $\frac{\pi}{2} - A$) opposite; that the rules

$$\begin{aligned} \sin a &= \tan b \tan \left(\tfrac{\pi}{2} - B\right) = \cos \left(\tfrac{\pi}{2} - c\right) \cos \left(\tfrac{\pi}{2} - A\right) \\ &= \tan b \cot B \qquad = \sin c \sin A, \end{aligned}$$

in which the mind associates *sine* and *middle*, *tangent* and *adjacent*, *cosine* and *opposite*, could not be forgotten or misapplied by the student. Such we should think, from the admirable and lucid manner in which the subject is presented, must be the author's opinion.

Our examination of these volumes confirms us in the high opinion we entertain of the author's series of text-books.

THE

MATHEMATICAL MONTHLY.

Vol. III. . . . JANUARY, 1861. . . . No. IV.

PRIZE PROBLEMS FOR STUDENTS.

I. $ABCD$ is a square, AC its diagonal, bisect AD in E, join BE cutting AC in F; then will

$$\frac{\triangle AEF}{1} = \frac{\triangle CEF}{2} = \frac{\triangle ABE}{3} = \frac{\triangle BCF}{4}.$$

II. If $\frac{a-b}{ay+bx} = \frac{b-c}{bz+cx} = \frac{c-a}{cy+az} = \frac{a+b+c}{ax+by+cz}$; then each of these fractions $= \frac{1}{x+y+z}$.

III. If d be the perpendicular, dropped from the angle C on the opposite side of a triangle, show that $\sin C = \frac{cd}{ab}$. Deduce the formula

$$\sin(A+B) = \sin A \cos B + \cos A \sin B.$$

IV. An hyperbola, having its asymptotes parallel to the axes of co-ordinates, may be made to pass through the angular points of the quadrilateral formed by the four lines whose equations are

(1) $y + x + 6a = 0$, (3) $y + 6x + a = 0$,
(2) $y + 2x + 3a = 0$, (4) $y + 3x + 2a = 0$.

V. Three lines, x, y, z, are drawn from the angular points of a triangle to a point within it. Show that when $x^n + y^n + z^n$ is a minimum we have

$$\frac{\sin(x,y)}{x^{n-1}} = \frac{\sin(x,z)}{y^{n-1}} = \frac{\sin(y,z)}{z^{n-1}},$$

in which (x, y) denotes the angle between x and y.

Solutions of these problems must be received by March 1, 1861.

NOTES AND QUERIES.

1. *To express the sines and cosines of the sum and difference of two angles in terms of the sines and cosines of the angles themselves.*

Let O be the origin of rectangular co-ordinates, and the pole of polar co-ordinates, the axis of x being the polar axis; then the square of the distance between two points expressed in polar co-ordinates (r, φ), (r', φ'), and in rectangular co-ordinates (x, y), (x', y'), is

$$(1) \qquad r^2 + r'^2 - 2rr' \cos(\varphi - \varphi') = (x - x')^2 + (y - y')^2.$$

But these two systems of co-ordinates are connected by the equations

$$r^2 = x^2 + y^2, \qquad x = r\cos\varphi, \qquad y = r\sin\varphi,$$
$$r'^2 = x'^2 + y'^2, \qquad x' = r'\cos\varphi', \qquad y' = r'\sin\varphi',$$

and therefore (1) reduces to

$$(2) \quad \cos(\varphi - \varphi') = \frac{x}{r}\cdot\frac{x'}{r'} + \frac{y}{r}\cdot\frac{y'}{r'} = \cos\varphi\cos\varphi' + \sin\varphi\sin\varphi'.$$

If we change the sign of φ', then $\sin\varphi'$, or y', will change its sign; therefore

$$(3) \quad \cos(\varphi + \varphi') = \frac{x}{r}\cdot\frac{x'}{r'} - \frac{y}{r}\cdot\frac{y'}{r'} = \cos\varphi\cos\varphi' - \sin\varphi\sin\varphi'.$$

From (1) we get

$$rr'\cos(\varphi - \varphi') = xx' + yy';$$
$$\therefore r^2r'^2 - r^2r'^2\cos^2(\varphi - \varphi') = r^2r'^2 - (xx' + yy')^2,$$
$$\therefore r^2r'^2\sin^2(\varphi - \varphi') = x^2y'^2 - 2xx'yy' + x'^2y^2,$$
$$\therefore rr'\sin(\varphi - \varphi') = xy' - yx',$$
$$\therefore \sin(\varphi - \varphi') = \frac{x}{r}\cdot\frac{y'}{r'} - \frac{y}{r}\cdot\frac{x'}{r'} = \sin\varphi\cos\varphi' - \cos\varphi\sin\varphi',$$

and $$\sin(\varphi + \varphi') = \frac{x}{r}\cdot\frac{y'}{r'} + \frac{y}{r}\cdot\frac{x'}{r'} = \sin\varphi\cos\varphi' + \cos\varphi\sin\varphi'.$$

2. *In a spherical triangle, to express the cosine of an angle in terms of sines and cosines of the sides.*

Let ABC be any spherical triangle, O the centre of the sphere, being the origin of co-ordinates, and OC the axis of z; let r denote the radius of the sphere, and (x, y, z), (x', y', z') the co-ordinates of A and B respectively; then the square of the chord AB gives

$$(1) \quad r^2 + r^2 - 2r^2 \cos AOB = (x - x')^2 + (y - y')^2 + (z - z')^2.$$

But these two systems of co-ordinates are connected by the equations

$$\begin{aligned} r^2 &= x^2 + y^2 + z^2, & r^2 &= x'^2 + y'^2 + z'^2, \\ x &= r \sin\theta \cos\varphi, & x' &= r \sin\theta' \cos\varphi', \\ y &= r \sin\theta \sin\varphi, & y' &= r \sin\theta' \sin\varphi', \\ z &= r \cos\theta, & z' &= r \cos\theta'; \end{aligned}$$

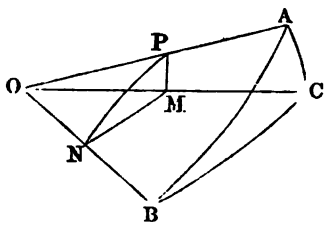
and (1) reduces to

$$\begin{aligned} \cos AOB &= \frac{x}{r}\cdot\frac{x'}{r} + \frac{y}{r}\cdot\frac{y'}{r} + \frac{z}{r}\cdot\frac{z'}{r}. \\ &= \sin\theta \sin\theta' \cos\varphi \cos\varphi' + \sin\theta \sin\theta' \sin\varphi \sin\varphi' + \cos\theta \cos\theta' \\ &= \sin\theta \sin\theta' (\cos\varphi \cos\varphi' + \sin\varphi \sin\varphi') + \cos\theta \cos\theta' \\ &= \sin\theta \sin\theta' \cos(\varphi - \varphi') + \cos\theta \cos\theta'; \end{aligned}$$

or $\cos c = \sin a \sin b \cos C + \cos a \cos b.$

The method of proof used in these notes is perfectly general, since all the equations used are universally true.

3. *Note on* NAPIER'S *Rules of Circular Parts.* — The formulas for the solution of right-angled spherical triangles are very easily deduced, as follows: Let the triangle ABC be right angled at C, and O be the centre of the sphere. Take any point, P, in AO; draw PM perpendicular to OC, MN perpendicular to OB, and join NP. Since the planes AOC and BOC are perpendicular to each other,



$$\therefore PN^2 = PM^2 + MN^2 = OP^2 - OM^2 + OM^2 - ON^2 = OP^2 - ON^2;$$

therefore PNO is a right angle. Hence

(1) $\frac{ON}{OP} = \frac{ON}{OM} \cdot \frac{OM}{OP}$; i. e. $\cos c = \cos a \cos b$.

(2) $\frac{PM}{OP} = \frac{PM}{PN} \cdot \frac{PN}{OP}$; i. e. $\sin b = \sin B \sin c$, $\sin a = \sin A \sin c$.

(3) $\frac{MN}{ON} = \frac{MN}{PN} \cdot \frac{PN}{ON}$; i. e. $\tan a = \cos B \tan c$, $\tan b = \cos A \tan c$.

(4) $\frac{PM}{OM} = \frac{PM}{MN} \cdot \frac{MN}{OM}$; i. e. $\tan b = \tan B \sin a$, $\tan a = \tan A \sin b$.

The last equations in (2), (3), (4), follow by a simple interchange of letters.

From the product of the second and third equations of (4) we get

$$\tan A \tan B = \frac{\tan a \tan b}{\sin a \sin b} = \frac{1}{\cos a \cos b} = \frac{1}{\cos c} \text{ by (1).}$$

(5) $$\therefore \cos c = \cot A \cot B.$$

Multiply crosswise the third equation of (2) and the second of (3); we obtain

$$\sin a \cos B \tan c = \tan a \sin A \sin c.$$

$$\therefore \cos B = \frac{\sin A \cos c}{\cos a} = \sin A \cos b \quad \text{by (1).}$$

(6) $$\therefore \cos B = \sin A \cos b, \qquad \cos A = \sin B \cos a.$$

Out of the five quantities a, b, c, A, B, only ten combinations of three can be formed; and, therefore, the ten equations just found solve all the cases of right-angled spherical triangles. Any two of the five quantities being given, and a third required, some one of these ten equations will give the required quantity in terms of the given ones.

These ten equations are comprised in the six following corresponding theorems, since four of them apply twice to the same triangle.

(1) cos hyp = product of cosines of sides.

(5) cos hyp = products of cotangents of angles.

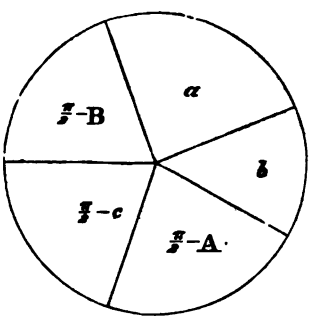
(2) sin side = sine of opposite angle × sin hyp.

(3) tan side = tan hyp × cos included angle.

(4) tan side = tan opposite angle × sine of other side.

(6) cos angle = cos opposite side × sine of other angle.

These six formulas may be condensed into two rules, known as NAPIER'S *Rules of Circular Parts*, which were first published by him in a work entitled *Mirifici Logarithmorum Canonis Descriptio*. Edinburgh, 1614.



Omitting the right angle, the circular parts are the two sides, including the right angle, and the complements of the hypothenuse and the two other angles; thus, a, b, $\frac{\pi}{2} - A$, $\frac{\pi}{2} - c$, $\frac{\pi}{2} - B$, are the five circular parts, and selecting any one of the five as the *middle part*, two of the remaining four are *adjacent*, and the other two *opposite* to the middle part. Then NAPIER'S two rules are

sine of middle part = product of tangents of adjacent parts.

sine of middle part = product of cosines of opposite parts.

These rules are memorized for use by associating *sine* and *middle*, *tangent* and *adjacent*, *cosine* and *opposite*.

The following table shows that these two rules involve the ten cases already found; the middle part stands first, the adjacent parts next, and the opposite parts in the third member of the equations.

$\sin a = \tan b \tan\left(\frac{\pi}{2} - B\right) = \cos\left(\frac{\pi}{2} - A\right) \cos\left(\frac{\pi}{2} - c\right)$, by Rules;

$\sin a = \tan b \cot B = \sin A \sin c$. Reduced.

$\sin b = \tan a \tan\left(\frac{\pi}{2} - A\right) = \cos\left(\frac{\pi}{2} - B\right) \cos\left(\frac{\pi}{2} - c\right)$, by Rules;

$\sin b = \tan a \cot A = \sin B \sin c$. Reduced.

$$\sin\left(\frac{\pi}{2}-A\right)=\tan b\tan\left(\frac{\pi}{2}-c\right)=\cos a\cos\left(\frac{\pi}{2}-B\right),\qquad\text{by Rules;}$$

$$\cos A=\tan b\cot c\qquad=\cos a\sin B.\qquad\text{Reduced.}$$

$$\sin\left(\frac{\pi}{2}-c\right)=\tan\left(\frac{\pi}{2}-A\right)\tan\left(\frac{\pi}{2}-B\right)=\cos a\cos b,\qquad\text{by Rules;}$$

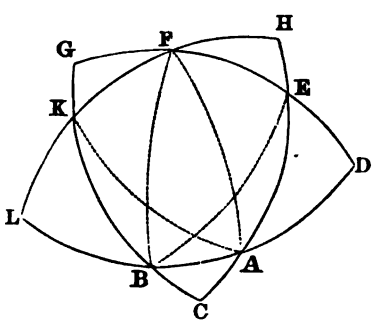
$$\cos c=\cot A\cot B\qquad=\cos a\cos b.\qquad\text{Reduced.}$$

$$\sin\left(\frac{\pi}{2}-B\right)=\tan a\tan\left(\frac{\pi}{2}-c\right)=\cos b\cos\left(\frac{\pi}{2}-A\right),\qquad\text{by Rules;}$$

$$\cos B=\tan a\cot c\qquad=\cos b\sin A.\qquad\text{Reduced.}$$

Besides this method of proving these rules, NAPIER has indicated another, on pp. 32–35 of the work already referred to, which is as follows: —

Let ABC be a spherical triangle right-angled at C. With A and B as poles describe great circles, and produce all the sides of ABC to meet them. The five triangles ABC, ADE, EHF, FGK, KLB are all right angled, and we propose to show that they all have the *same circular parts*. Let us first consider ABC and ADE. Denote the elements of ABC, as usual, by a, b, c, A, B; then, in the triangle ADE, the angle $EAD=A$; the side AD is the complement of AB, and equals $\frac{\pi}{2}-c$; the side DE equals $\frac{\pi}{2}-B$, because it measures the angle EBD, which is the complement of $ABC=B$; the angle DEA equals $\frac{\pi}{2}-a$, because it is the complement of the angle BEC, which is measured by $BC=a$; the side EA is the complement of AC, and equals $\frac{\pi}{2}-b$. The elements of ADE are therefore $\frac{\pi}{2}-c$, $\frac{\pi}{2}-B$, $\frac{\pi}{2}-b$, $\frac{\pi}{2}-a$, A, and its circular parts are $\frac{\pi}{2}-c$, $\frac{\pi}{2}-B$, b, a, $\frac{\pi}{2}-A$, which are those of ABC.



It can be shown in the same way that the triangles AED and EHF have the same circular parts, and so on.

The following table exhibits in one view all the elements of these five allied triangles, beginning with the hypothenuse, and numbering around to the right.

	ABC	ADE	EHF	FGK	KLB
1.	$AB=c$	$AE=\frac{\pi}{2}-b$	$FE=B$	$KF=A$	$BK=\frac{\pi}{2}-a$
2.	$BAC=A$	$DEA=\frac{\pi}{2}-a$	$HEF=c$	$GKF=\frac{\pi}{2}-b$	$LBK=B$
3.	$CA=b$	$DE=\frac{\pi}{2}-B$	$HF=\frac{\pi}{2}-A$	$GK=a$	$LB=\frac{\pi}{2}-c$
4.	$BC=a$	$AD=\frac{\pi}{2}-c$	$EH=b$	$FG=\frac{\pi}{2}-B$	$KL=\frac{\pi}{2}-A$
5.	$ABC=B$	$EAD=A$	$HEF=\frac{\pi}{2}-a$	$KFG=c$	$LKB=\frac{\pi}{2}-b$

The complements of the 3d, 4th, 5th, and 1st elements of any one of these triangles are the 1st, 2d, 3d, and 4th elements of the right-hand adjacent triangle, while their 2d and 5th elements are the same; and by applying this law to the five triangles in succession, we shall come back to the set of elements from which we started.

If we apply theorems (1) and (5) to each of these five triangles, or either of the remaining four theorems twice to each one of them, or all the theorems to any one of them, we shall obtain the ten original equations.

It follows, therefore, since the circular parts of these triangles are the same, that we obtain the same results by applying the two *Rules of Circular Parts* to each of the five circular parts of the same triangle, as by applying the theorems to the allied triangles, either all to one, or two to each of the five in succession.

This method considers each rule, not as the statement of dissimilar properties of one triangle, but as the statement of similar properties of five allied triangles. These notes on Spherical Trigonometry are substantially from TODHUNTER'S excellent work on this subject.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. XII., Vol. II.

The first Prize is awarded to Asher B. Evans, Madison Univ., N. Y.

The second prize is not awarded.

The third prize is awarded to John M. Terrill, Linneus, Mo.

Prize Solution of Problem I.

By Miss H. S. Hazeltine, Worcester, Mass.

If AB, CD be chords of a circle at right angles to each other, prove that the sum of the arcs AC, BD is equal to half the circumference.

Designate by E the point where the chords intersect each other, and draw the chord AC. In the right triangle AEC, the sum of the acute angles is 90°; whence

$$\text{arc } BC + \text{arc } DD = \text{semi-circumference} = \text{arc } AC + \text{arc } BD,$$

or $180° - (\text{arc } AC + \text{arc } BD)$, according as E falls outside or inside of the circle.

Prize Solution of Problem II.

By John M. Terrill, Linneus, Mo.

Prove that the cube of any number and the number itself, being divided by 6, leave the same remainder.

Take $a = 6b + c$. Then $a^3 = 216\,b^3 + 108\,b^2 c + 18\,b c^2 + c^3$. In dividing this last expression by 6, if there be a remainder it must be found in the last term, c^3, because all the other terms are multiples of 6. This remainder is found by trial to be equal to c for all values of the latter less than 6, that is, for all possible values of c; that is, when c is 1, 2, 3, 4, or 5, the remainders of c^3 divided by 6 are 1, 2, 3, 4, 5.

Second Solution.

By Miss Mary A. Willis, Nunda, N. Y.

We know (Math. Monthly, Vol. II., No. XI.) that $x^3 - x$ is divisible by 6 when x is a whole number. Therefore, since $x^3 = 6q + r$

and $x = 6q' + r'$, in which q, q' are the quotients, and r, r' the remainders, the equation

$$x^3 - x = 6(q - q') + r - r'$$

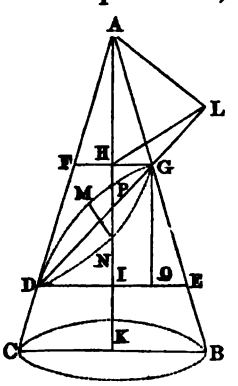
must be divisible by 6; and therefore $r - r'$ is divisible by 6, which cannot be the case, since $r - r' < 6$. Therefore $r - r' = 0$, or $r = r'$, as was to be proved.

Prize Solution of Problem III.

By Asher B. Evans, Madison University, Hamilton, N. Y.

It is required to divide a given right cone into two parts, having the ratio of m to n, by a plane making an angle of p degrees with the axis.

Draw FG and DE parallel to the base BC and cutting the axis of the cone in H and I; draw AL perpendicular to DG produced, and join H and L; then, if MN be the diameter, conjugate to DG in the elliptical section $GMDN$, MN will be equal to $\sqrt{(DE \cdot FG)}$. The part ADG, above the elliptical section is an oblique elliptical cone, and



$$\frac{AL}{3} \cdot DG \cdot MN\frac{\pi}{4} = \frac{\pi}{12} AL \cdot DG\sqrt{(DE \cdot FG)} = \text{its solidity.}$$

The solidity of the whole cone ABC is

$$\frac{\pi}{12} BC^2 \cdot AK;$$

therefore, by the question,

$$(1) \qquad m + n : m :: BC^2 \cdot AK : AL \cdot DG \cdot \sqrt{(DE \cdot FG)}$$

The triangles DEG and AHL are similar; for, if a circle be described on AG as diameter, the points H and L will be in the circumference, and $AHL = AGL = DGE$, and $HLA = HGA = DEG$; hence

$$(2) \qquad DG \cdot AL = DE \cdot AH.$$

Combining equations (1) and (2)

(3) $\qquad m+n : m :: BC^2 . AK : DE . AH\sqrt{(DE . FG)}.$

Let $DE = 2x$, $FG = 2y$, $BC = 2b$, $AK = ab$; then $AI = ax$, $AH = ay$; and (3) becomes

$$m+n : m :: 4ab^3 : 4axy\sqrt{(xy)}. \qquad \therefore\ xy = b^2\left(\frac{m}{m+n}\right)^{\frac{1}{2}}.$$

Again, $DPI = DGQ = p$, and $GQ = a(y-x)$ and $DQ = x+y$.

$$\therefore\ a\tan p\,(y-x) = x+y; \quad \text{and} \quad x = b\left(\frac{m}{m+n}\right)^{\frac{1}{2}}\sqrt{\left(\frac{a\tan p - 1}{a\tan p + 1}\right)},$$

$$y = b\left(\frac{m}{m+n}\right)^{\frac{1}{2}}\sqrt{\left(\frac{a\tan p + 1}{a\tan p - 1}\right)}.$$

No satisfactory solution for the cases in which the section is a parabola or hyperbola has been received.

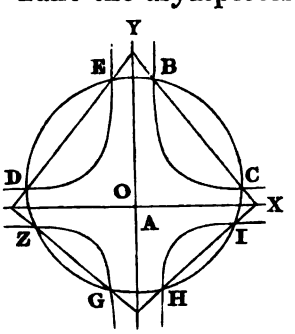
Problem IV. is pronounced to be wrong by several of the competitors.

Prize Solution of Problem V.

By Asher B. Evans, Madison University, Hamilton, N. Y.

Describe an ellipse having its foci at any points on one of the asymptotes of an equilateral hyperbola. The ellipse will cut each branch of the hyperbola and its conjugate in two real or imaginary points. Prove that a chord joining one pair of these points will be perpendicular to one of the other similar chords. Prove it first for a circle and then for an ellipse.

Take the asymptotes Ax and Ay as axes; and let x_1y_1, x_2y_2, x_3y_3, x_4y_4 represent B, C, Z, G, the four points common to the hyperbola and the circle; also let θ and θ_1 represent the angles which BC and GZ make with the axis of abscissas; then $\frac{y_2-y_1}{x_2-x_1} = \tan\theta$, and $\frac{y_4-y_3}{x_4-x_3} = \tan\theta_1$. The equation of the hyperbola is



(1) $\qquad xy = c^2;$

that of the circle is

$$x^2 + (y-a)^2 = r^2. \tag{2}$$

Combining (1) and (2),

$$y^4 - 2ay^3 + (a^2 - r^2)y^2 + c^4 = 0. \tag{3}$$

The roots of (3) are y_1, y_2, y_3, y_4; hence, by the theory of equations,

$$y_1 y_2 y_3 y_4 = c^4. \tag{4}$$

The co-ordinates of B, C, Z, G must satisfy (1); this condition gives $\tan\theta = -\frac{c^2}{y_1 y_2}$, and $\tan\theta_1 = -\frac{c^2}{y_3 y_4}$.

$$\therefore \tan\theta \tan\theta_1 = \frac{c^4}{y_1 y_2 y_3 y_4} = 1,$$

because (4) exists. Hence $\sin\theta \sin\theta_1 = \cos\theta \cos\theta_1$,

or $$\sin\theta \sin\theta_1 - \cos\theta \cos\theta_1 = \cos(\theta + \theta_1) = 0.$$

$$\therefore \theta + \theta_1 = 90°.$$

If E, D, I, H be four points common to the conjugate hyperbola and the circle, it is evident from considerations of symmetry that ED and IH, BC and ZG, make equal angles with the axis of x, whenever BC and DE are at right angles respectively with HI and ZG. By substituting the equation of the ellipse for that of the circle in the above proof, it may similarly be shown that this property does *not* hold for the ellipse.

No solution of the problem as amended (see Editorial Items for November number) has been received.

Some of the competitors are requested to take more pains with their penmanship and the arrangement of their solutions.

Simon Newcomb,
W. P. G. Bartlett,
Truman Henry Safford.

EXPOSITION OF THE PROCESS OF MATHEMATICAL DEVELOPMENT.

By JOHN PATERSON, A. M., Albany, N. Y.

[Concluded from Page 61.]

12. THE synchronous and successive series of forces for the time $\psi\, 1_t$ are therefore $1\,\varphi^{IV}$, $1\,\psi\,\varphi'''$, $1\,\psi^2\,\varphi''$, $1\,\psi^3\,\varphi'$, $1\,\psi^4\,\varphi^\circ$, and $1\,\varphi^{IV}$, $2\,\psi\,\varphi'''$, $3\,\psi^2\,\varphi''$, $4\,\psi^3\,\varphi'$, $1\,\psi^4\,\varphi^\circ$. Applying the successive forces to a succeeding interval $h\,1_t$ of time, and reasoning as above with $\psi\,1_t$, we get the following:—

$\psi\,1_t$	$h\,1_t$.			
$1\,\varphi^{IV} = 1\,\varphi^{IV}$:	$\frac{1.2}{1}\,h\,\varphi'''$,	$\frac{1.2.3}{1.2}\,h^2\,\varphi''$,	$\frac{1.2.3.4}{1.2.3}\,h^3\,\varphi'$,	$\frac{1.2.3.4.1}{1.2.3.4}\,h^4\,\varphi^\circ = 1\,h^4\,\varphi^\circ$.
$1\,\psi\,\varphi''' = 2\,\psi\,\varphi'''$:	$\frac{2.3}{1}\,\psi\,h\,\varphi''$,	$\frac{2.3.4}{1.2}\,\psi\,h^2\,\varphi'$,	$\frac{2.3.4.1}{1.2.3}\,\psi\,h^3\,\varphi^\circ = 4\,\psi\,h^3\,\varphi^\circ$.	
$1\,\psi^2\,\varphi'' = 3\,\psi^2\,\varphi''$:	$\frac{3.4}{1}\,\psi^2\,h\,\varphi'$,	$\frac{3.4.1}{1.2}\,\psi^2\,h^2\,\varphi^\circ = 6\,\psi^2\,h^2\,\varphi^\circ$.		
$1\,\psi^3\,\varphi' = 4\,\psi^3\,\varphi'$:	$\frac{4.1}{1}\,\psi^3\,h\,\varphi^\circ = 4\,\psi^3\,h\,\varphi^\circ$.			
$1\,\psi^4\,\varphi^\circ = 1\,\psi^4\,\varphi^\circ$.				

Thus $(\psi^4 + 4\,\psi^3\,h + 6\,\psi^2\,h^2 + 4\,\psi\,h^3 + h^4)\,\varphi^\circ = (\psi + h)^4\,\varphi^\circ$ is the final result generated by the force $1\,\varphi^{IV}$ in the time $(\psi + h)\,1_t$.

13. So far the exposition has treated of free operations only, no restriction whatever being applied to the generating forces, and the result is the normal form of development. The transition to other forms, such as $(\psi \pm h)^{\pm n}$, $(\psi \pm h)^{\pm \frac{p}{q}}$, $a^{\pm \psi}$, $e^{\pm \sqrt{-1}\psi}$, etc., etc., is effected by the introduction of a suitable condition in each case at the outset of the process.

14. Forces of the first order φ', originating from different sources, and entering into combination, produce effects or phenomena adapted to observation, the admeasurement of some simple forms of which may be the subject of a future article; and the present one will conclude with an application of the preceding principles to the demonstration of a result hitherto obtained by a different method.

Let $n = 2$: then the force $1\,\varphi''$ is an independent primitive generator of the first order and second degree, agreeing in character with a *constant force* as usually denominated, such as is the force of gravity at the surface of the earth. This force $1\,\varphi''$, entering into combination with a reacting force of the same order and degree, which, in the case of the terrestrial force mentioned, exists in the form of a heavy body, for which 1_μ may be written as denoting a unit of mass, will generate force of the second order and first degree φ', in a first interval of time $\psi'\,1_t$, in such amount that, with its ultimate value remaining constant, φ' will produce m times the effect in the equal subsequent interval $\psi''\,1_t$ of time, that was generated by it in the time $\psi'\,1_t$ of its own genesis: this force of the first degree φ', in this case, is the velocity (the unit mark of which may be 1_λ) accruing to the body acted upon by the force of gravity; and the effect of this velocity is the distance described by the falling body, the unit mark of which may be 1_l, and which here takes the place of the force zero φ°.

The constant force $1\,\varphi''$ generates velocity uniformly, the ultimate value of which is $m\,1_\lambda$ at the expiration of the time $\psi'\,1_t$: it is therefore $\frac{m}{2}.1_\lambda$ at the expiration of the time $\frac{1}{2}'.\psi'\,1_t$, with which value it would carry the body 1_μ the distance $\frac{m}{2}\psi^2 1_l$ in the time $\psi''\,1_t$, and consequently does carry it the distance $\frac{m}{4}.\psi^2 1_l$ in the time $\frac{1}{2}''.\psi'\,1_t$, and the mth part of this distance in time $\frac{1}{2}'.\psi'\,1_t$; which again is the distance 1_μ will be carried by the increment of velocity generated by $1\,\varphi''$ in the second half-interval of time $\frac{1}{2}''.\psi'\,1_t$. This brings the expression $\left(\frac{1}{4}+\frac{m}{4}+\frac{1}{4}\right)\psi^2\,1_\lambda$; which, when $\psi = 1$, gives the equation $\left(\frac{1}{4}+\frac{m}{4}+\frac{1}{4}\right)1_\lambda = 1.1_\lambda$, because the effect is to be units in the first unit of time, and the effect is a measure of its cause for that time. The equation gives $m = 2$, and we construct the dynamical triangle: —

$$\frac{\psi\, 1_t.}{1\,\varphi''};\quad \frac{\hbar\, 1_t.}{\frac{1.2}{1}\hbar\, 1_\lambda,\ \frac{1.2.1}{1.2}\hbar^2\, 1_t.}$$

$$2\,\psi\,.\,1_\lambda;\quad \frac{2.1}{1}\,\psi\,\hbar\, 1_t.$$

$$1\,\psi^2\,.\,1_t.$$

The total is $(1\,\psi^2 + 2\,\psi\,\hbar + 1\,\hbar^2)\,1_t = (\psi + \hbar)^2\,1_t$, the square of the time.

But when the unit of time $1_t = 1$ second, and the unit of distance $1_l = 1$ inch, observation shows that the distance described by any body under the influence of gravity in the first unit of time is approximately 193 inches: then the coefficient of gravity will not be units, but 193, when the unit of time is one second, and the unit of distance is one inch; and the result becomes $(1\,\psi^2 + 2\,\psi\,\hbar + \hbar^2)$ 193 inches in the time $(\psi + \hbar)\,1'$.

15. The preceding paragraphs have shown the genesis of a series of forces or powers $2\,\psi\,\varphi^{n-1}$, $3\,\psi^2\,\varphi^{n-2}$, $4\,\psi^3\,\varphi^{n-3}$, $\ldots\,(n-1)\,\psi^{n-2}\varphi''$, $n\,\psi^{n-1}\,\varphi'$, $1\,\psi^n\,\varphi^\circ$, by the primitive power $1\,\varphi^n$, in the time $\psi\,1_t$. The number n is integral and indefinite, and consequently the series may be prolonged *ad libitum* to the right, or in the positive direction; thus placing $1\,\psi^n\varphi^\circ$, the zero of force or power, the final phenomenal result, at any required distance below the primitive independent $1\,\varphi^n$. The numerical coefficients, the exponents of the time ψ, and the indices of the powers φ, all traverse the series by unit steps. Setting aside the *efficient* symbol φ, transition from any term to the next lower one is made by adding unity both to the numerical factor and to the exponent of the literal factor of the *coefficient* of the term selected; or indeed any term whatever is reached at once by adding or subtracting the number appropriate to the interval of transition.

16. Proceeding from right to left in the series (A), and subtracting any convenient number (say 7) from the coefficient and exponent

of the several terms, there arises the series (B), in which terms may be continued in both directions.

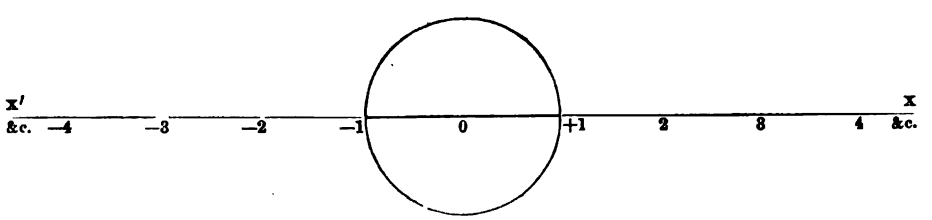
(A) $0\psi^{-1}$, $1\psi^{0}$, $2\psi^{1}$, $3\psi^{2}$, $4\psi^{3}$, $5\psi^{4}$, $6\psi^{5}$, $7\psi^{6}$, $8\psi^{7}$, $9\psi^{8}$, $10\psi^{9}$, $11\psi^{10}$, $12\psi^{11}$, $13\psi^{12}$, &c.

(B) &c. $-6\psi^{-7}$, $-5\psi^{-6}$, $-4\psi^{-5}$, $-3\psi^{-4}$, $-2\psi^{-3}$, $-1\psi^{-2}$, $0\psi^{-1}$, $1\psi^{0}$, $2\psi^{1}$, $3\psi^{2}$, $4\psi^{3}$, $5\psi^{4}$, $6\psi^{5}$, &c.

(C) &c. $1\psi^{-7}$, $1\psi^{-6}$, $1\psi^{-5}$, $1\psi^{-4}$, $1\psi^{-3}$, $1\psi^{-2}$, $1\psi^{-1}$, $1\psi^{0}$, $1\psi^{1}$, $1\psi^{2}$, $1\psi^{3}$, $1\psi^{4}$, $1\psi^{5}$, &c.

The series (B) is merely the series (A) prolonged also in the negative direction, and is therefore the general form of the series expressed in *successive* measures, to the terms of which correspond the *synchronous* measures expressed in the series (C). The coefficient 0 shows that $1\,\varphi^{n}$ is not generated, but must be assumed; and indeed $1\,\varphi^{n}$ is the unique primitive independent power, or *constant* generating force in mathesis, all beyond zero in (B) proceeding by indefinite regression towards negative infinity. The term $0\,\psi^{-1}$, which has $1\,\psi^{-1}\varphi^{\circ}$ for the measure of its synchronous value, marks and masks the transition from positive to negative coefficients; for when $\psi = 1$, the coefficient of the term $\frac{1}{1}\,\psi^{\circ}$ expresses the ratio of two equals, while $0\,.\,\frac{1}{1}\,\varphi^{n}$ indicates equality and opposition in the directions of the two forces which respectively generate the numerator and denominator. This transition from positivity to negativity, and *vice versa*, is effected by passing from linear through angular magnitude, to the opposite direction in space, by means of the revolving radius unity or 1_{r}, and comprises the region of logarithms and exponentials, circulars and elliptics, and many definite integrals, etc.

For the time $\psi\,1_{i} = 1\,.\,1_{i}$, the following diagram exhibits the measures of the coefficient in (B):—



17. That the relations of the terms of (B) are not changed by transition through the critical value, or point of discontinuity, 0, is shown by forming proportions between any two pairs of terms, as thus: —

$$5\,\psi^{4} : 11\,\psi^{10} :: \quad 3\,\psi^{3} : 9\,\psi^{8},\ 14\,\psi^{12} = 14\,\psi^{12};$$

$$-2\,\psi^{-3} : \ 4\,\psi^{3} :: -4\,\psi^{-5} : 2\,\psi,\quad 0\,\psi^{-2} = \ 0\,\psi^{-2};$$

$$-2\,\psi^{3} : \ 4\,\psi^{8} :: \quad 3\,\psi^{8} : 9\,\psi^{6},\ \ 7\,\psi^{5} \ = \ 7\,\psi^{5};$$

where the identical equations are deduced by adding together respectively the coefficients and the exponents of the means and of the extremes. From this it is inferred that under like operations, results deduced respectively from terms on the left and on the right of the zero in (B) are equally true.

18. The power zero φ° may be placed anywhere in the series (B), and then its coefficient must always be replaced by the corresponding term in (C). If φ° be placed to the right of the term $0\,\psi^{-1}$, the example always has an independent primitive $1\,\varphi^{n}$; for instance, the coefficient $3\,\psi^{3}$ corresponds to $1\,\psi^{3}\varphi^{\circ}$ as phenomenal result, and to $1\,\varphi''$ as independent primitive (which is a case furnished by nature in the constant force of gravitation). If $2\,\psi\,\varphi^{\circ}$ were selected, it corresponds to $1\,\psi\,\varphi^{\circ}$; that is, there is no force $2\,\psi\,\varphi$ generated, but the result of the genesis is $\psi\,\varphi^{\circ} = \psi\,1_{l}$, 1_{l} being the linear unit, or measure of the effect of a unit 1_{λ} of constant velocity in the unit of time 1_{t}, the independent primitive of which is $1\,\varphi' = 1_{\lambda}$ = a power of the first order. If $1\,\psi^{0}\varphi^{\circ}$ be selected, then $1\,\varphi^{\circ} = 1_{l}$ is not generated, but is an assumed unit of magnitude.

By these steps, it is seen that the most convenient method of counting will be to place φ° in (C), and, passing diagonally one term to the left in (B), proceed to the primitive $1\,\varphi^{n}$. Thus $1\,\psi^{3}\varphi^{\circ}$ recalls the generators $2\,\psi\,\varphi'$ and $1\,\varphi''$; $1\,\psi\,\varphi^{\circ}$ recalls $1\,\varphi'$; but while

$1\,\psi^0\,\varphi^\circ = 1\,\varphi^\circ$ may be assumed, it has also some kind of power $0\,\psi^{-1}\varphi$ for its generator; and as 0 is a measure of the ratio of equality, or a logarithm which expresses the number of times that one quantity is greater than another, its interpretation will appear in another place: $0\,\psi^{-1} = \frac{d\psi}{\psi} = d\,.\log\psi$.

The term $1\,\psi^{-1}\varphi^\circ$ calls up the series

$$-1\,\psi^{-2}\varphi',\ -2\,\psi^{-3}\varphi'',\ -3\,\psi^{-4}\varphi''',\ \ldots.$$

ad infinitum; and any term further to the left will evoke a similar series of negative coefficients and exponents, with efficients ascending indefinitely upwards from units in order, always commencing in (B) with φ' one step to the left of the term φ° selected in (C).

19. The scheme for the development of the function $(\psi + h)^n$ is as follows: —

$\psi\,1_\iota$: $h\,1_\iota$.

$1\,\varphi^n$; $\frac{1.2}{1}h\varphi^{n-1}$, $\frac{1.2.3}{1.2}h^2\varphi^{n-2}$, $\frac{1.2.3.4}{1.2.3}h^3\varphi^{n-3}$, $\frac{1.2.3.4.5}{1.2.3.4}h^4\varphi^{n-4}$, &c.

$2\psi\,\varphi^{n-1}$; $\frac{2.3}{1}\psi\,h\,\varphi^{n-2}$, $\frac{2.3.4}{1.2}\psi\,h^2\varphi^{n-3}$, $\frac{2.3.4.5}{1.2.3}\psi\,h^3\varphi^{n-4}$, &c.

$3\psi^2\varphi^{n-2}$; $\frac{3.4}{1}\psi^2\,h\,\varphi^{n-3}$, $\frac{3.4.5}{1.2}\psi^2\,h^2\varphi^{n-4}$, &c.

$4\psi^3\varphi^{n-3}$; $\frac{4.5}{1}\psi^3\,h\,\varphi^{n-4}$, &c.

$5\psi^4\varphi^{n-4}$; &c.

&c.

On reaching the power $\varphi^{n-n} = \varphi^\circ$, all the terms of the lower diagonal series (the hypothenuse of the arithmetical triangle) are to be divided by the last factor of the numerators to get the final result.

Let $n = 4$: the triangle becomes

$\psi 1_t$.	$h 1_t$.			
$1\ \varphi^{IV}$;	$\frac{1.2}{1} h \varphi'''$,	$\frac{1.2.3}{1.2} h^2 \varphi''$,	$\frac{1.2.3.4}{1.2.3} h^3 \varphi'$,	$\frac{1.2.3.4.1}{1.2.3.4} h^4 \psi^\circ = 1\ h^4 \varphi^\circ$.
$2\ \psi \varphi'''$;	$\frac{2.3}{1} \psi h \varphi''$,	$\frac{2.3.4}{1.2} \psi h^2 \varphi'$,	$\frac{2.3.4.1}{1.2.3} \psi h^3 \varphi^\circ = 4\ \psi h^3 \varphi^\circ$.	
$3\ \psi^2 \varphi''$;	$\frac{3.4}{1} \psi^2 h \varphi'$,	$\frac{3.4.1}{1.2} \psi^2 h^2 \varphi^\circ = 6\ \psi^2 h^2 \varphi^\circ$.		
$4\ \psi^3 \varphi'$;	$\frac{4.1}{1} \psi^3 h \varphi^\circ = 4\ \psi^3 h \varphi^\circ$.			
$1\ \psi^4 \varphi^\circ$.				

Which is $(\psi^4 + 4\psi^3 h + 6\psi^2 h^2 + 4\psi h^3 + h^4)\varphi^\circ = (\psi + h)^4 \varphi^\circ$.

During the past time $\psi 1_t$, the independent primitive $1\varphi^{IV}$ of the fourth order has generated the final result $1\psi^4\varphi^\circ$, or has caused the unit of mass 1_μ to describe the distance $1\psi^4 1_l$ (or has generated the distance $1\psi^4 1_l$); and has also generated the forces or powers $4\psi^3\varphi'$, $3\psi^2\varphi''$, and $2\psi\varphi'''$, respectively of the first, second, and third orders, which become constant generators at the expiration of the time $\psi 1_t$. The power $4\psi^3\varphi'$ is a constant velocity $4\psi^3 1_\lambda$, which will of itself generate the distance $4\psi^3 h 1_l$ in the future time $h 1_t$: it is the only velocity in existence in the system at the commencement of the time $h 1_t$, and is the differential coefficient of ψ^4, which gives the motion in the tangent of the angle that the curve makes with the axis of abscissa at the point ψ, rectangular co-ordinates understood. The power $3\psi^2\varphi''$ is a constant force which generates velocity uniformly; and as it has itself generated the velocity $4\psi^3 1_\lambda$ during its growth, it now adds the distance $6\psi^2 h^2 1_l$ to the foregoing terms, by means of the velocity $12\psi^2 h 1_\lambda$ which it generates in the time $h 1_t$. The power $2\psi\varphi'''$ is a constant force of the third order, which generates force of the second order uniformly, which generates velocity increasingly during the time $h 1_t$, and adds the distance $4\psi h^3 1_l$ to the foregoing terms. And lastly the primitive power $1\varphi^{IV}$ itself generates the distance $h^4 1_l$ in the time $h 1_t$, just as the distance $\psi^4 1_l$ was its final result in the time $\psi 1_t$.

20. Positive development begins at unity, proceeds towards positive infinity, and can be arrested at any term. Conversely, negative development begins at negative infinity, but may be taken up at any term, and concludes at unity. A closing portion of the scheme follows: —

$\psi\, 1_t$.	$h\, 1_t$.			
&c.	&c.	&c.	&c.	
$-7\,\psi^{-8}\,\varphi^{\mathrm{VII}}$;	$\frac{-7!}{-5!}\psi^{-8}h\,\varphi^{\mathrm{VI}}$,	$\frac{-7!}{-4!\,2!}\psi^{-8}h^2\,\varphi^{\mathrm{V}}$, ...	$\frac{-7!\,1}{7!}\psi^{-8}h^7\,\varphi^{\circ}$.	
$-6\,\psi^{-7}\,\varphi^{\mathrm{VI}}$;	$\frac{-6!}{-4!}\psi^{-7}h\,\varphi^{\mathrm{V}}$,	$\frac{-6!}{-3!\,2!}\psi^{-7}h^2\,\varphi^{\mathrm{IV}}$, ...	$\frac{-6!\,1}{6!}\psi^{-7}h^6\,\varphi^{\circ}$.	
$-5\,\psi^{-6}\,\varphi^{\mathrm{V}}$;	$\frac{-5!}{-3!}\psi^{-6}h\,\varphi^{\mathrm{IV}}$,	$\frac{-5!}{-2!\,2!}\psi^{-6}h^2\,\varphi'''$, ...	$\frac{-5!\,1}{5!}\psi^{-6}h^5\,\varphi^{\circ}$.	
$-4\,\psi^{-5}\,\varphi^{\mathrm{IV}}$;	$\frac{-4!}{-2!}\psi^{-5}h\,\varphi'''$,	$\frac{-4!}{-1.2!}\psi^{-5}h^2\,\varphi''$,	$\frac{-4!}{3!}\psi^{-5}h^3\,\varphi'$,	$\frac{-4!\,1}{4!}\psi^{-5}h^4\,\varphi^{\circ}$.
$-3\,\psi^{-4}\,\varphi'''$;	$\frac{(-3)\,(-2)}{1}\psi^{-4}h\,\varphi''$,	$\frac{-3!}{1.2}\psi^{-4}h^2\,\varphi'$,	$\frac{-3!\,1}{3!}\psi^{-4}h^3\,\varphi^{\circ}$.	
$-2\,\psi^{-3}\,\varphi''$;	$\frac{(-2)\,(-1)}{1}\psi^{-3}h\,\varphi'$,	$\frac{(-2)\,(-1)}{1}\psi^{-3}h^2\,\varphi^{\circ}$.		
$-1\,\psi^{-2}\,\varphi'$;	$\frac{(-1)\,1}{1}\psi^{-2}h\,\varphi^{\circ}$.			
$+1\,\psi^{-1}\,\varphi^{\circ}$;				

in which the sign $n!$, or $\pm\frac{2n+1}{2}!$, as usual denotes the continued product $1.2....n$; or $\frac{1}{2}.\frac{3}{2}...\frac{2n+1}{2}$; or $(-\frac{1}{2})\,(-\frac{3}{2})...\left(-\frac{2n+1}{2}\right)$

The first diagonal line

$$\left(\frac{1}{\psi}-\frac{1}{\psi^2}h+\frac{1}{\psi^3}h^2-\frac{1}{\psi^4}h^3+\frac{1}{\psi^5}h^4-\frac{1}{\psi^6}h^5+\frac{1}{\psi^7}h^6-\frac{1}{\psi^8}h^7\right)\varphi^{\circ}$$

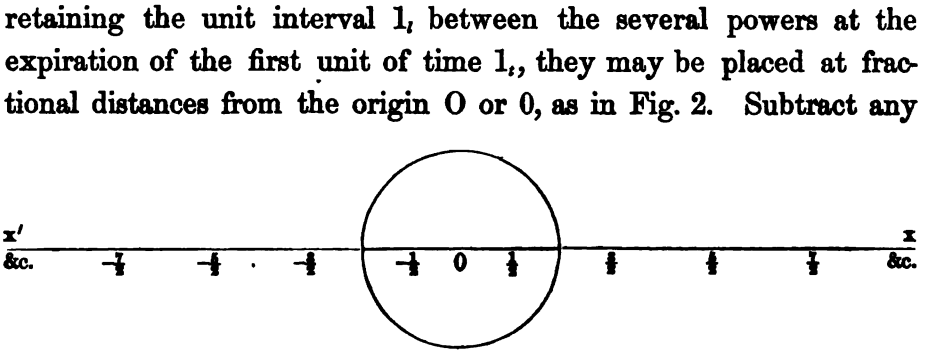
is the development of $(\psi+h)^{-1}$ carried through eight terms. But here, as in the case of positive development, the power zero φ° may be placed on any term of the first perpendicular line; where the

corresponding diagonal line, being divided throughout by the numerical factor common to every term, will express the development of $(\psi + h)$ to the index of ψ in the first term. For example, place φ° upon the fourth term from the last, and there appears: —

$\psi\, 1_t.$	$h\, 1_t.$			
&c.	&c.	&c.	&c.	
$-7\,\psi^{-8}\varphi^{IV};$	$\frac{-7\,!}{-5\,!}\psi^{-8}h\varphi'''$,	$\frac{-7\,!}{-4\,!\;2\,!}\psi^{-8}h^2\varphi''$,	$\frac{-7\,!}{-3\,!\;3\,!}\psi^{-8}h^3\varphi'$,	$\frac{-7\,!\,1}{-3\,!\;4\,!}\psi^{-8}h^4\varphi^\circ.$
$-6\,\psi^{-7}\varphi''';$	$\frac{-6\,!}{-4\,!}\psi^{-7}h\varphi''$,	$\frac{-6\,!}{-3\,!\;2\,!}\psi^{-7}h^2\varphi'$,	$\frac{-6\,!\,1}{-3\,!\;3\,!}\psi^{-7}h^3\varphi^\circ.$	
$-5\,\psi^{-6}\psi'';$	$\frac{(-5)\,(-4)}{1}\psi^{-6}h\varphi'$,	$\frac{(-5)(-4)\,1}{1\,.\,2}\psi^{-6}h^2\varphi^\circ.$		
$-4\,\psi^{-5}\varphi';$	$\frac{(-4)\,1}{1}\psi^{-5}h\varphi^\circ.$			
$+1\,\psi^{-4}\varphi^\circ.$				

Total $\left(\frac{1}{\psi^4} - 4\frac{h}{\psi^5} + 10\frac{h^2}{\psi^6} - 20\frac{h^3}{\psi^7} + 35\frac{h^4}{\psi^8}\right)\varphi^\circ = (\psi + h)^{-4}\varphi^\circ$ to five terms.

21. To pass from integral to fractional development, observe, that retaining the unit interval 1_t between the several powers at the expiration of the first unit of time 1_t, they may be placed at fractional distances from the origin O or 0, as in Fig. 2. Subtract any



fractional number $\frac{p}{q}$, say $\frac{5}{2}$, from the coefficients and exponents of the terms of the successive integral series (B), and there arises the successive fractional series (D), to which corresponds the synchronous series (E): —

(B) &c. $-4\psi^{-6}$, $-3\psi^{-4}$, $-2\psi^{-3}$, $-1\psi^{-2}$, $0\psi^{-1}$, $1\psi^{0}$, $2\psi^{1}$, $3\psi^{2}$, $4\psi^{3}$, $5\psi^{4}$, $6\psi^{5}$, &c.

(D) &c. $-\frac{1}{2}\psi^{-\frac{1}{2}}$, $-\frac{1}{2}\psi^{-\frac{1}{2}}$, $-\frac{3}{2}\psi^{-\frac{1}{2}}$, $-\frac{5}{2}\psi^{-\frac{7}{2}}$, $-\frac{5}{2}\psi^{-\frac{7}{2}}$, $-\frac{5}{2}\psi^{-\frac{9}{2}}$, $-\frac{1}{2}\psi^{-\frac{3}{2}}$, $\frac{1}{2}\psi^{-\frac{1}{2}}$, $\frac{3}{2}\psi^{\frac{1}{2}}$, $\frac{5}{2}\psi^{\frac{3}{2}}$, $\frac{7}{2}\psi^{\frac{5}{2}}$, $\frac{9}{2}\psi^{\frac{7}{2}}$, &c.

(E) &c. $1\psi^{-\frac{1}{2}}$, $1\psi^{-\frac{1}{2}}$, $1\psi^{-\frac{1}{2}}$, $1\psi^{-\frac{3}{2}}$, $1\psi^{-\frac{7}{2}}$, $1\psi^{-\frac{9}{2}}$, $1\psi^{-\frac{3}{2}}$, $1\psi^{-\frac{1}{2}}$, $1\psi^{\frac{1}{2}}$, $1\psi^{\frac{3}{2}}$, $1\psi^{\frac{5}{2}}$, &c.

1°. Place the zero of power at the term $1\psi^{\frac{1}{2}}$, and get the triangle:

$\psi\, 1_t.$	$h\, 1_t.$			
&c.	&c.	&c.	&c.	
$-\frac{1}{2}\psi^{-\frac{1}{2}}\varphi^{IV};$	$\frac{-\frac{3}{2}!}{-\frac{1}{2}.1}\psi^{-\frac{1}{2}}h\varphi'''$,	$\frac{-\frac{3}{2}!}{2!}\psi^{-\frac{1}{2}}h^2\varphi''$,	$\dots\ \frac{-\frac{3}{2}!\frac{1}{2}.1}{4!}\psi^{-\frac{1}{2}}h^4\varphi^\circ = -\frac{1.1.3.5}{1.2.3.4}(\frac{1}{2})^4\frac{h^4}{\psi^{\frac{7}{2}}}\varphi^\circ.$	
$-\frac{1}{2}\psi^{-\frac{1}{2}}\varphi'''$;	$\frac{-\frac{3}{2}!}{1}\psi^{-\frac{1}{2}}h\varphi''$,	$\frac{-\frac{3}{2}!\frac{1}{2}}{2!}\psi^{-\frac{1}{2}}h^2\varphi'$,	$\frac{-\frac{3}{2}!\frac{1}{2}.1}{3!}\psi^{-\frac{1}{2}}h^3\varphi^\circ = +\frac{1.1.3}{3!}(\frac{1}{2})^3\frac{h^3}{\psi^{\frac{5}{2}}}\varphi^\circ.$	
$-\frac{1}{2}\psi^{-\frac{1}{2}}\varphi'';$	$\frac{-\frac{1}{2}.\frac{1}{2}}{1}\psi^{-\frac{1}{2}}h\varphi'$,	$\frac{-\frac{1}{2}.\frac{1}{2}.1}{2!}\psi^{-\frac{1}{2}}h^2\varphi^\circ = -\frac{1.1}{2!}(\frac{1}{2})^2\frac{h^2}{\psi^{\frac{3}{2}}}\varphi^\circ.$		
$+\frac{1}{2}\psi^{-\frac{1}{2}}\varphi';$	$\frac{\frac{1}{2}.1}{1}\psi^{-\frac{1}{2}}h\varphi^\circ = +\frac{1}{1}(\frac{1}{2})\frac{h}{\psi^{\frac{1}{2}}}\varphi^\circ.$			
$1\psi^{\frac{1}{2}}\varphi^\circ.$				

Which is the development of $+\sqrt{(\psi+h)}$ to five terms.

2°. Place the power zero at the term $1\psi^{-\frac{1}{2}}$, and get

$\psi\, 1_t.$	$1\, h_t.$			
&c.	&c.	&c.	&c.	
$-\frac{1}{2}\psi^{-\frac{3}{2}}\varphi^{IV};$	$\frac{-\frac{7}{2}!}{-\frac{3}{2}!}\psi^{-\frac{3}{2}}h\varphi'''$,	$\dots\ \frac{-\frac{7}{2}!}{3!}\psi^{-\frac{3}{2}}h^3\varphi'$,	$\frac{-\frac{7}{2}!1}{4!}\psi^{-\frac{3}{2}}h^4\varphi^\circ = +\frac{1.3.5.7}{4!}(\frac{1}{2})^4\frac{h^4}{\psi^{\frac{9}{2}}}\varphi^\circ.$	
$-\frac{5}{2}\psi^{-\frac{7}{2}}\varphi'''$;	$\frac{-\frac{5}{2}!}{-\frac{1}{2}.1}\psi^{-\frac{7}{2}}h\varphi''$,	$\dots\ \frac{-\frac{5}{2}!}{2!}\psi^{-\frac{7}{2}}h^2\varphi'$,	$\frac{-\frac{5}{2}!1}{3!}\psi^{-\frac{7}{2}}h^3\varphi^\circ = -\frac{1.3.5}{3!}(\frac{1}{2})^3\frac{h^3}{\psi^{\frac{7}{2}}}\varphi^\circ.$	
$-\frac{3}{2}\psi^{-\frac{5}{2}}\varphi'';$	$\frac{-\frac{3}{2}!}{1}\psi^{-\frac{5}{2}}h\varphi'$,	$\frac{-\frac{3}{2}!1}{2!}\psi^{-\frac{5}{2}}h^2\varphi^\circ = +\frac{1.3}{1.2}.(\frac{1}{2})^2\frac{h^2}{\psi^{\frac{5}{2}}}\varphi^\circ.$		
$-\frac{1}{2}\psi^{-\frac{3}{2}}\varphi';$	$\frac{-\frac{1}{2}.1}{1}\psi^{-\frac{3}{2}}h\varphi^\circ = -\frac{1}{1}(\frac{1}{2})\frac{h}{\psi^{\frac{3}{2}}}\varphi^\circ.$			
$1\psi^{-\frac{1}{2}}\varphi^\circ.$				

Which is the development of $-\sqrt{(\psi+h)}$ to five terms.

The development of $(\psi+h)^{\frac{1}{2}}$ is found as follows: —

$\psi 1_t$.	$h 1_t$.
&c.	&c. &c.
$-\frac{7}{2}\psi^{-\frac{1}{2}}\varphi^{\text{VIII}}$;	$\frac{-\frac{7}{2}!}{-\frac{1}{2}!}\psi^{-\frac{1}{2}}h\,\varphi^{\text{VII}}, \ldots \frac{-\frac{7}{2}!\,\frac{7}{2}!\,1}{8!}\psi^{-\frac{1}{2}}h^8\varphi^\circ = +\frac{7.5.3.1.1.3.5.7}{8!}(\frac{1}{2})^8\frac{h^8}{\psi^{\frac{1}{2}}}\varphi^\circ.$
$-\frac{5}{2}\psi^{-\frac{7}{2}}\varphi^{\text{VII}}$;	$\frac{-\frac{5}{2}!}{-\frac{1}{2}.1}\psi^{-\frac{7}{2}}h\,\varphi^{\text{VI}}, \ldots \frac{-\frac{5}{2}!\,\frac{7}{2}!\,1}{7!}\psi^{-\frac{7}{2}}h^7\varphi^\circ = -\frac{7.5.3.1.1.3.5}{7!}(\frac{1}{2})^7\frac{h^7}{\psi^{\frac{7}{2}}}\varphi^\circ.$
$-\frac{3}{2}\psi^{-\frac{5}{2}}\varphi^{\text{VI}}$;	$\frac{-\frac{3}{2}!}{1}\psi^{-\frac{5}{2}}h\,\varphi^{\text{V}}, \ldots \frac{-\frac{3}{2}!\,\frac{7}{2}!\,1}{6!}\psi^{-\frac{5}{2}}h^6\varphi^\circ = +\frac{7.5.3.1.1.3}{6!}(\frac{1}{2})^6\frac{h^6}{\psi^{\frac{5}{2}}}\varphi^\circ.$
$-\frac{1}{2}\psi^{-\frac{3}{2}}\varphi^{\text{V}}$;	$\frac{-\frac{1}{2}.\frac{1}{2}}{1}\psi^{-\frac{3}{2}}h\varphi^{\text{IV}}, \ldots \frac{-\frac{1}{2}.\frac{7}{2}!\,1}{5!}\psi^{-\frac{3}{2}}h^5\varphi^\circ = -\frac{7.5.3.1.1}{5!}(\frac{1}{2})^5\frac{h^5}{\psi^{\frac{3}{2}}}\varphi^\circ.$
$+\frac{1}{2}\psi^{-\frac{1}{2}}\varphi^{\text{IV}}$;	$\frac{\frac{1}{2}!}{1}\psi^{-\frac{1}{2}}h\,\varphi''', \ldots \frac{\frac{7}{2}!\,1}{4!}\psi^{-\frac{1}{2}}h^4\varphi^\circ = +\frac{7.5.3.1}{4!}(\frac{1}{2})^4\frac{h^4}{\psi^{\frac{1}{2}}}\varphi^\circ.$
$+\frac{3}{2}\psi^{\frac{1}{2}}\varphi'''$;	$\frac{\frac{1}{2}!}{\frac{1}{2}.1}\psi^{\frac{1}{2}}h\,\varphi'', \quad \frac{\frac{7}{2}!}{\frac{1}{2}.2!}\psi^{\frac{1}{2}}h^2\varphi', \quad \frac{\frac{7}{2}!\,1}{\frac{1}{2}.3!}\psi^{\frac{1}{2}}h^3\varphi^\circ = +\frac{7.5.3}{3!}(\frac{1}{2})^3\psi^{\frac{1}{2}}h^3\varphi^\circ.$
$+\frac{5}{2}\psi^{\frac{3}{2}}\varphi''$;	$\frac{\frac{7}{2}!}{\frac{3}{2}!}\psi^{\frac{3}{2}}h\,\varphi', \quad \frac{\frac{7}{2}!\,1}{\frac{3}{2}!\,2!}\psi^{\frac{3}{2}}h^2\varphi^\circ = +\frac{7.5}{1.2}(\frac{1}{2})^2\psi^{\frac{1}{2}}h^2\varphi^\circ.$
$+\frac{7}{2}\psi^{\frac{5}{2}}\varphi'$;	$\frac{\frac{7}{2}.1}{1}\psi^{\frac{5}{2}}h\,\varphi^\circ = +\frac{7}{1}(\frac{1}{2})\psi^{\frac{1}{2}}h\varphi^\circ.$
$1\,\varphi^{\frac{7}{2}}\varphi^\circ$.	

Thus positive fractional development is not arrested, but passes over zero by revolution.

22. When h is negative, the future time $h 1_t$, after the expiration of the time $\psi 1_t$, is changed for past time $-h 1_t$, the results of which are to be subtracted from those of the full time $\psi 1_t$. The factors of the numerators, even when under the negative sign, always tend in the positive direction; those of the denominators also advance for time *future* to $\psi 1_t$; consequently they will reverse their direction for time that is *past* when $\psi 1_t$ has expired. Then for the development of $(\psi - h)^4$, the form will be

$\psi 1_t$.	$h 1_t$.
$1\,\varphi^{\text{IV}}$;	$\frac{2!}{-1}h\,\varphi''', \quad \frac{3!}{-2!}h^2\varphi'', \quad \frac{4!}{-3!}h^3\varphi', \quad \frac{4!\,1}{-4!}h^4\varphi^\circ = +1\,h^4\varphi^\circ.$
$2\,\psi^3\varphi'''$;	$\frac{3!}{-1}\psi\,h\,\varphi'', \quad \frac{4!}{-2!}\psi\,h^2\varphi', \quad \frac{4!\,1}{-3!}\psi\,h^3\varphi^\circ = -4\,\psi\,h^3\varphi^\circ.$
$3\,\psi^2\varphi''$;	$\frac{4!}{-1.2!}\psi^2\,h\,\varphi', \quad \frac{4!\,1}{-2!\,2!}\psi^2\,h^2\varphi^\circ = +6\,\psi^2\,h^2\varphi^\circ.$
$4\,\psi^3\varphi'$;	$\frac{4.1}{-1}\psi^3\,h\,\varphi^\circ = -4\,\psi^3\,h\,\varphi^\circ.$
$1\,\psi^4\varphi^\circ$.	

And for $(\psi - h)^{-4}$, the form will be

$\psi 1_t$.	$h 1_t$.		
&c.	&c.	&c.	&c.
$-7\,\psi^{-8}\varphi^{IV}$;	$\frac{-7\,!}{-5\,!\,-1}\psi^{-8}h\varphi'''$,	$\frac{-7\,!}{-4\,!\,-2\,!}\psi^{-8}h^2\varphi''$, ...	$\frac{-7\,!\,1}{-3\,!\,-4\,!}\psi^{-8}h^4\varphi^\circ = +\frac{35h^4}{\psi^8}\varphi^\circ$.
$-6\,\psi^{-7}\varphi'''$;	$\frac{-6\,!}{-3\,!\,-1}\psi^{-7}h\varphi''$,	$\frac{-6\,!}{-3\,!\,-2\,!}\psi^{-7}h^2\varphi'$,	$\frac{-6\,!\,1}{-2\,!\,-3\,!}\psi^{-7}h^3\varphi^\circ = +\frac{20h^3}{\psi^7}\varphi^\circ$.
$-5\,\psi^{-6}\varphi''$;	$\frac{-5\,!}{-3\,!\,-1}\psi^{-6}h\varphi'$,	$\frac{-5\,!\,1}{-3\,!\,-2\,!}\psi^{-6}h^2\varphi^\circ = +\frac{10h^2}{\psi^6}\varphi^\circ$.	
$-4\,\psi^{-5}\varphi'$;	$\frac{-4\,.\,1}{-1}\psi^{-5}h\varphi^\circ = +\frac{4h}{\psi^5}\varphi^\circ$.		
$+1\,\psi^{-4}\varphi^\circ$			

When the exponent of ψ is negative, uniform semicircular rotation commences with the beginning of the time $\psi 1_t$; when the coefficient of h is negative, an equal rotation begins with the time $h 1_t$.

NOTES ON PROBABILITIES.

By Simon Newcomb, Cambridge, Mass.

The problem with which the article on probabilities, in the Mathematical Monthly for May, 1860, was closed, belongs to a class which has been much discussed. We shall here repeat it, and then show a variation in it which leads to a result, which, at first sight, seems paradoxical.

An urn contains 100 balls, 1 of which is black, and 99 white; an individual, who falsifies one time in ten, draws a ball, and asserts that it is black. What is the probability that he tells the truth? We have found that it is $\frac{1}{12}$.

But suppose that the balls are numbered from 1 to 100, and that he asserts that the ball No. 79 is drawn. What is the probability that he tells the truth?

The solution is exhibited in tabular form, as follows. The first column contains the separate hypotheses which it is possible to make, column p gives the *a priori* probabilities of these hypotheses, and column q the probability of the observed event on the several hypotheses, as explained in the number of the Mathematical Monthly above referred to.

c	p	q	pq
No. 79 drawn	$\frac{1}{100}$	$\frac{9}{10}$	$\frac{9}{1000}$
Some other No. drawn,	$\frac{99}{100}$	$\frac{1}{990}$	$\frac{1}{1000}$

Whence the probability that No. 79 was drawn, or that the truth is told, is, by § 24, $\frac{\frac{9}{9000}}{\frac{9}{1000}+\frac{1}{1000}} = \frac{9}{10}$.

The value of q, on the second hypothesis, is obtained as follows: — Suppose that some other number than 79 is drawn. Then, if he is going to falsify, he has the 99 numbers which were not drawn, to choose from. Since there is supposed to be no reason why he should say 79 rather than any number, the probability that he will select this number from the 99 not drawn is $\frac{1}{99}$. But the probability that he will falsify in the first place is $\frac{1}{10}$. Whence the probability of saying 79 falsely is $\frac{1}{990}$.

Now the apparent paradox is, that the probability that he has told the truth is so vastly different when he says that a single black ball has been drawn from 100, from what it is when he says that the single number 79 has been so drawn, being $\frac{1}{12}$ in the first case, and $\frac{9}{10}$ in the second. Several forms of explanation may be given, and the reader can select that which seems to him most clear.

1. The difference of the results proceeds from the circumstance that he is supposed to have but two colors to choose from if he falsifies, viz. white and black, while he has 100 numbers. Since a white ball is drawn 99 times out of a hundred, and since he falsifies about one time in ten, he will have said black *about* 11 times in

every hundred, ten times falsely, and once truly. Whence I conclude that when he says black he does not tell the truth, for if I believed him to tell the truth every time, I should have to believe that an excessively improbable number of black balls had been drawn; instead of which I knew beforehand that he would make some ten or twelve false statements respecting the drawing of the black ball. But if 100 numbered balls have been drawn, his ten falsifications are supposed to have been divided amongst the 100 numbers I know not how, whence the chance that any one number was called falsely is only $\frac{1}{10}$.

If, however, I found that some one number, such as 79, had been called 10 or 12 times, I should conclude that the witness was disposed to falsify by saying 79, and thus I would not believe him, and the second case would be reduced to the first. Again, if there were a hundred colors from which he might choose, I would believe him when he said black, because I should conclude that if he were going to falsify, he would probably select some of the other 99 colors, and thus the first case of the problem would become identical with the second.

The following is an interesting problem, and might be useful. A bag contains an unknown number of tickets, numbered from one upwards. A person draws a ticket, notes the number, and replaces it; then repeats the operation till he has drawn s times. What judgment can he form as to the number of tickets in the bag?

Let p_n represent the *a priori* probability that the bag contains just n tickets, and no more. Let i be the largest number drawn in the s drawings. The number of tickets, then, cannot be less than i. We need not know any of the drawn numbers except the largest, because after we know this, every combination of smaller numbers will be equally probable on every admissible hypothesis, and will therefore be of no assistance in judging of these hypotheses. We

may then be supposed to know that s tickets have been drawn, and that the largest number observed is i. Now let us make our hypotheses. Suppose that the bag contains just i tickets. The probability that No. i would *not* be drawn on any one drawing is $\frac{i-1}{i}$. Hence the probability that it would not be drawn at all is $\frac{(i-1)^s}{i^s}$; hence the probability that it would be drawn on one or more trials is $\frac{i^s-(i-1)^s}{i^s}$. Again, if we suppose the bag to contain $i+h$ tickets, the probability that i would be the largest number drawn would be equal to the product of the probability that none of the h numbers above i would be drawn into the probability that i would be drawn if none of the h were drawn. The first of these probabilities is $\frac{i^s}{(i+h)^s}$; whence the probability that i would be the largest number is $\frac{i^s-(i-1)^s}{(i+h)^s}$. We can now arrange the quantities as before. We put for brevity $i^s-(i-1)^s=\triangle$.

c	p	q	pq
i balls	p_i	$\frac{\triangle}{i^s}$	$\frac{\triangle p_i}{i^s}$
$(i+1)$ balls	p_{i+1}	$\frac{\triangle}{(i+1)^s}$	$\frac{\triangle p_{i+1}}{(i+1)^s}$
$(i+2)$ balls	p_{i+2}	$\frac{\triangle}{(i+2)^s}$	$\frac{\triangle p_{i+2}}{(i+2)^s}$
&c.	&c.	&c.	&c.

Put P_n for the final probability that the bag contains just n tickets, and we have, by the general formula,

$$(1) \quad P_{i+h} = \frac{\triangle p_{i+h}}{(i+h)^s \left\{ \frac{\triangle p_i}{i^s} + \frac{\triangle p_{i+}}{(i+1)^s} + \&c. \right\}} = \frac{p_{i+h}}{(i+h)^s \cdot \Sigma}.$$

putting, for brevity,

$$\Sigma = \frac{p_i}{i^s} + \frac{p_{i+1}}{(i+1)^s} + \frac{p_{i+2}}{(i+2)^s} + \cdots$$

By taking the sum of a sufficient number of the values of P_{i+h} from $h=0$ upwards to make $\frac{1}{2}$, we should find that number which

it was an even chance whether the balls would fall short of, or exceed. In getting the value of Σ, the series extends to all the numbers which it is possible that the bag should contain.

As a particular case, suppose that all numbers up to a certain limit are, before the drawing, equally probable. The values of p_n will then be all equal; and by putting

$$S = \frac{1}{i^s} + \frac{1}{(i+1)^s} + \frac{1}{(i+2)^s} + \&c.,$$

for all possible values, we shall have

$$(2) \qquad P_i = \frac{1}{Si^s}, \quad P_{i+1} = \frac{1}{S(i+1)^s}, \ldots P_{i+k} = \frac{1}{S(i+k)^s}.$$

We see from these expressions that the highest number drawn is the most probable of *any single number*, so that if we were required to guess the exact number contained in the bag, we ought to guess this one. But if i and k are not very small, several numbers above i will each be almost as probable as i itself, and the guess would therefore probably be wrong. But it would be more satisfactory to find the probability that the number is contained between given limits. Thus, the probability that it does not exceed $i+k$ is, by summing the separate probabilities,

$$(3) \qquad \frac{1}{S}\left\{\frac{1}{i^s} + \frac{1}{(i+1)^s} + \frac{1}{(i+2)^s} + \ldots + \frac{1}{(i+k)^s}\right\}.$$

As a numerical example, suppose that all numbers are equally before drawing, and that upon drawing four times, the highest number drawn is 10. Then we have $i=10$, $s=4$, $k=\infty$.

Also $$S = \sum_1^\infty{}_x \frac{1}{x^4} - \sum_1^9{}_x \frac{1}{x^4} = \frac{\pi^4}{90} - \sum_1^9{}_x \frac{1}{x^4} = 0.000386650,$$

$$\frac{1}{i^s} = 0.0001 = \frac{1}{10^4}, \quad \frac{1}{(i+1)^s} = 0.0000683, \quad \frac{1}{(i+2)^s} = 0.0000482.$$

The sum of the first two numbers is rather more than $\frac{1}{2}S$, and

the sum of the first three rather greater. It is therefore rather less than an even chance that the bag contains 11 tickets, or less, but rather more than an even chance that it does not contain more than 12. If we make $k = 10$ in (3), the resulting value of the probability is $\frac{8}{9}\frac{4}{5}\frac{8}{6}\frac{1}{6}$. We might therefore wager more than 8 against 1 that the bag did not contain more than 20 balls.

As another numerical example, suppose that I read in a paper the addresses of ten persons, all at the same post-office, but having different boxes, of which the numbers are given. The highest number was 200. What is the probable number of boxes which the office contains, supposing that one box is as likely to be let to any person as another?

To solve this, we observe that the sum of the series

$$1 + \frac{1}{2^{10}} + \frac{1}{3^{10}} + \ldots + \frac{1}{(x-1)^{10}}$$

is

$$\frac{\pi^{10}}{93555} - \frac{1}{9x^9} - \frac{1}{2x^{10}} - \frac{10}{12x^{11}} + \frac{132}{72x^{12}} - \&c.$$

The sum from x to infinity is, therefore,

$$\frac{1}{x^{10}} + \frac{1}{(x+1)^{10}} + \frac{1}{(x+2)^{10}} + \ldots = \frac{1}{9x^9} + \frac{1}{2x^{10}} + \frac{10}{12x^{11}} - \frac{11}{6x^{12}} + \&c.$$

and by taking the difference of these values for different values of x, we can find the sum of the series between finite limits. Representing by $\Sigma_{i,i'}$ the sum of the series between the limits i and i' inclusive, we have

$$\Sigma_{200,\infty} = 22194, \qquad \Sigma_{200,215} = 11110,$$

in units of the twenty-sixth place of decimals. There is, therefore, about an even chance that the post-office does not contain more than 215 boxes.

When i is so large that we may neglect all terms after the first in the expressions for the sum of the series, we shall have

$$S = \Sigma_{i\infty} = \frac{1}{(S-1)\,i^{s-1}} \qquad \Sigma_{i,i'} = \frac{1}{(S-1)\,i^{s-1}} - \frac{1}{(S-1)\,(i'-1)^{s-1}};$$

which, from equation (3), gives for the probability that the number required is contained between the limits i and i' inclusive,

$$P_{i,i'} = 1 - \frac{i^{s-1}}{(i'+1)^{s-1}}.$$

If we wish to find the limits within which there is a given probability that the number is contained, we obtain from the last equation,

$$i' = \frac{i}{(1-P_{i,i})^{\frac{1}{s-1}}} - 1.$$

Mathematical Monthly Notices.

A Treatise on Astronomy, Descriptive, Theoretical, and Physical, Designed for Schools, Academies, and Private Students. By H. N. Robinson, A. M., Formerly Professor of Mathematics in the United States Navy; Author of a Treatise on Arithmetic, Algebra, Geometry, Trigonometry, Surveying, Calculus, Natural Philosophy, &c., &c. New York: Ivison & Phinney. 1858.

This treatise professes to give an exposition, not only of the operations of practical astronomy, but of the motions of the heavenly bodies as deduced from the law of gravitation.

A few extracts from the work would be amply sufficient to enable the professional astronomer to judge of the author's success in carrying out his plan, but as teachers and students generally will wish to form such a judgment, we shall add such comments as the extracts suggest.

On page 67 it is stated that, "The shape of the earth, thus ascertained by actual measurement, is just what theory would give to a body of water equal to our globe, and revolving on an axis in 24 hours." Unfortunately, the shape of our earth is very far indeed from being that which theory would give to a body of water "equal to our globe, and revolving on an axis in 24 hours," as any one will find by putting into numbers the results given by any writer on the figure of a homogeneous revolving fluid mass in equilibrium; e.g. De Pontécoulant, Système du Monde, Vol. II.

On page 79 and 80 the method given for computing the relation between the horizontal parallax and semidiameter of the moon is entirely erroneous in theory, from the *tangents* of those quantities being used instead of the *sines*. Practically, the mistake is of slight moment.

On page 89 we find the statement that the real velocity of the earth in the orbit "varies in the inverse ratio as the radius vector; *or it varies directly as the apparent diameter of the sun.*" Here the author has forgotten that the earth does not always move in a direction at right an-

gles to that of the line drawn through the earth and the sun; and his statement is not true, unless the velocity is first multiplied by the cosine of this angle.

The following explanation will strike the physicist as being both unique and original:—

"The inquiry is sometimes made why we do not have the warmest weather about the summer solstice, and the coldest weather about the time of the winter solstice.

"This would be the case if the sun immediately ceased to give extra warmth, on arriving at the summer solstice; but if it could radiate extra heat to warm the earth three weeks before it came to the solstice, it would give the same extra heat three weeks after; and the northern portion of the earth must continue to increase in temperature as long as the sun continues to radiate more than its medium degree of heat over the surface, at any particular place. Conversely, the whole region of country continues to grow cold as long as the sun radiates less than its mean annual degree of heat over that region. The medium degree of heat, for the whole year, and for all places, of course, takes place when the sun is on the equator; the average temperature, at the time of the two equinoxes. The medium degree of heat, for our northern summer, considering only two seasons in the year, takes place when the sun's declination is about 12 degrees north; and the medium degree of heat, for winter, takes place when the sun's declination is about 12 degrees south; and if this be true, the heat of summer will begin to decrease about the 20th of August, and the cold of winter must essentially abate on or about the 16th of February, in all northern latitudes." —p. 100.

We find some difficulty in making the different parts of this explanation seem consistent with each other; but it can hardly be necessary to tell any intelligent teacher of Natural Philosophy that no part of the explanation has any reference to the real cause of the phenomenon. May we ask how the author's theory would explain the fact that at the depth of twenty or thirty yards the temperature is highest in December and lowest in June?

In showing how to compute the equation of time, the author does not state whether the true or mean longitude of the sun is to be used in calculating that part of the equation of time which proceeds from "the variable inclination of the sun's motion to the equator." From the first explanation of the figure we should suppose he meant S, in fig. 22, p. 103, to represent the position of the mean sun; but he after speaks of ΥS as representing the sun's longitude, from which we might infer that he meant the true longitude to be used in finding the second correction.

On page 116 we find the statement that a planet appears stationary when moving nearly toward the earth, the author entirely forgetting that in the case supposed the planet would change its apparent position in consequence of the motion of the earth.

From page 142: "It [Mercury] is rarely visible, owing to its small size and proximity to the sun, and it never appears larger to the naked eye than a star of the fifth magnitude." Any one who has ever seen Mercury must know that it shines with nearly or quite the brilliancy of a star of the first magnitude; and it is frequently visible at its elongations, provided one takes the proper opportunity, and is not in too high a latitude.

On page 190 directions are given for finding the radius of the earth from the horizontal parallax of the moon, and the statement is made that by this means we can find the diameter of the earth to a greater degree of exactness than by any one actual measurement. The moon's horizontal parallax cannot be determined by observation to within a second of arc, that is, within about $\frac{1}{3600}$ of its whole value. Its cube is therefore subjected to an uncertainty of $\frac{1}{1200}$ of its whole value; and since the formula on p. 191 virtually contains the cube of the parallax as a factor, the final result is uncertain to the $\frac{1}{1200}$ part of its whole amount from this cause alone; and the diameter of the earth thus found may therefore be in error by six or seven miles. Yet it is said that "this result is more to be relied on than any other." We have said nothing about the uncertainty in the force of gravity, and the mass of the moon, both of which enter, or ought to enter, into the formula.

In § 186, pages 209 and 210, we read:—

"From the earliest ages the eccentricity of the orbit [of the earth] has been diminishing; and this diminution will probably continue until it is annihilated altogether, and the orbit becomes a circle; after which it will open out in another direction, again become eccentric, and increase in eccentricity to a certain moderate amount, and then again decrease.

"The period for these *vibrations*,' though calculable, *has never been calculated* further than to satisfy us that it is not to be reckoned by hundreds or even by thousands of years.' It is a period so long that the history of astronomy, and of the whole human race, is but a point in comparison."

To this we reply, that it is a well-known fact that neither the eccentricity of the orbit of the earth, or of that of any of the planets, will continue to diminish till the orbit becomes circular; and it is moreover a demonstrable fact that the orbits of none of the planets will ever become mathematically perfect circles. The statement in the second paragraph is made in the face of the fact that the periods, not exactly of "these *vibrations*," but of the actual vibrations, have been repeatedly computed and their reciprocals published by LAGRANGE, in the "Berlin Memoirs;" by JACOBI, in his "Mathematische Werke;" by PONTÉCOULANT, in the third volume of his "Théorie Analytique du Système du Monde;" and by LEVERRIER, in the "Annales de l'Observatoire," Tom. II., and in the French "Connaissance des Temps."

On the very first page of the chapter devoted to planetary perturbations our author states that, "The direct effect of any of these bodies [the planets] on the position of the sun is absolutely insensible."

Now the actual fact is, that the effects of the attraction of the planets on the sun is always nearly as great as on each other, and sometimes so much greater that all the principal terms proceed from this cause.

"Students in Astronomy very naturally infer that the attractions of Venus and Jupiter (when those planets are nearest to the earth) affect the longitude of the moon. Indeed, some have been so confident of this as to infer that some of the lunar equations were the results of such planetary attractions. But we assure them it is not so; none of the lunar equations refer to any other disturbing body than the sun. Yet all other bodies do disturb the lunar motion, but the amount of other disturbing forces is absolutely insensible, as we shall show by the following analysis:" —

The result arrived at is, that the moon's longitude is not affected by either Venus or Jupiter by more than the "very small fraction of a second;" and the discussion, about an octavo page in length, closes with, "Therefore the perturbations produced on the moon's motion by the planets are not, and need not be, taken into account." (*Lunar Perturbations: A Fragment*, pp. 351, 352.)

Now the facts are, that some ten years ago the Astronomer Royal of England found from observation, and PROF. HANSEN from theory, that Venus produces an inequality of 20″ or more in the longitude of the moon; and in HANSEN'S *Lunar Tables* the sum of the terms depending on Venus may amount to nearly 40″, and even a few terms depending upon Mars and Jupiter, though small, are retained. It will be seen, by any one who will take the trouble to examine PROF. PEIRCE'S *Tables of the Moon*, that the effect of Venus is not omitted in the Lunar Ephemeris, printed in the *American Ephemeris and Nautical Almanac*. The author supposes that the problem of perturbations is solved by simply computing the intensity of the force at any instant; and we do not find even an intimation that this force is a function of the time, and that an integration is necessary to determine what change it may produce in the lapse of centuries. Indeed, the discussion shows only the first faint conceptions of the problem. But we must pass over, for want of space, numerous other and equally flagrant and unpardonable errors into which the author has fallen. That the author of a text-book on Astronomy should be so unacquainted with the modern history and present state of the science is most extraordinary; and that he should misapprehend the character of its problems is hardly less so. We do not contend that the author of an elementary treatise should be a profound and original investigator in the science of which he treats; but we do insist that he should be able to write with ordinary accuracy; that he should have an accurate knowledge of the present state of the

science; and that he should have a clear comprehension of the general principles upon which the science is based. We regret to say that our author has lamentably failed in all these necessary particulars.

Introductory Course of Natural Philosophy, for the use of Schools and Academies. Edited from GANOT'S Popular Physics, by WILLIAM G. PECK, A. M., Professor of Mathematics, Columbia College, New York. Published by A. S. Barnes & Burr, 51 and 53 John Street, New York. 1860. pp. 480. 308 Illustrations.

During the past few years several excellent works on Physics have appeared in France. These works are marked by a good arrangement of the subjects treated, scientific accuracy, and clearness of statement combined with a large number of illustrations, which for beauty of design and execution have never been equalled. They are more particularly described by the term *experimental*, as distinguished from those in which the subjects are treated mathematically. On account of the acknowledged excellence of these works much use has been made of the matter and illustrations by authors preparing works on experimental Physics for the use of teachers and students in this country. Among the latest and best of them is one entitled *Cours de Physique, purement experiméntale, a l'usage des gens du monde, des aspirantes au brevet supérieur des élèves des écoles normales, des institutions de demoiselles, et en général, des personnes étrangères aux connaissances mathématiques, par* A. GANOT, Professeur de Physique; ouvrage orné de 308 magnifiques vignettes dessinées par GAGNIET, gravées sur bois par QUARTELEY, DUJARDIN, LAMBERT, et en relief, sur cuivre, par E. SALLE. We have given the title in full, to show that the author intended the work, not only for schools of certain grades, but *for the use of everybody;* and on this account, and so far as possible to avoid the necessity of apparatus, every principle is fully illustrated with copper-plate engravings far surpassing in design and execution anything ever before introduced into a text-book. We have never seen a *strictly popular work* on Physics, or Natural Philosophy, at all to be compared with it, and we sincerely thank the Editor and Publishers of the American edition for making the work more accessible to teachers and students in this country. The editor has made judicious use of the original work, and without producing an exact translation, has preserved its symmetry and better adapted it to the educational wants of this country; and the Publishers by illustrating their edition with *fac-simile* copies of the original engravings, by special arrangement with the author, M. GANOT, will have the satisfaction of knowing by its use that their efforts are appreciated. We are glad to see that the applications of fluids, steam, &c., as motive powers have not been introduced. Such an extension would have made the work too large and expensive for general use, and would not be in harmony with the plan of simply developing and explaining elementary principles. We take great pleasure in recommending this work to the attention of teachers and pupils, as well as the general reader, as being in all respects by far the best elementary one upon the subject which we have examined.

ERRATA. — Vol. III., p. 11, l. 7, for "where," read "whose"; p. 12, l. 3, for "or," read "and"; p. 14, l. 18, dele "even" after when; p. 16, last line, for $\frac{n(n-1)}{1.2} a^{n-1} x'$, read "$n a^{n-1} x'$;" p. 17, l. 10, for $\frac{1}{16a}$ read $\frac{1}{16a^2}$.

THE

MATHEMATICAL MONTHLY.

Vol. III. ... FEBRUARY, 1861. ... No. V.

PRIZE PROBLEMS FOR STUDENTS.

I. Find a point without two concentric circles, from which if two tangents be drawn to the circles the one shall be the double of the other.

II. Any whole number and its fifth power, when divided by 30, leave the same remainder; also, the number and its seventh power, when divided by 42, leave the same remainder. — Communicated by Asher B. Evans, Madison University, Hamilton, N. Y.

III. There are n straight lines making, with another fixed straight line, the angles $\alpha, \beta, \gamma \ldots$; a point, P, is taken, such that the sum of the squares of the perpendiculars from it on these n lines is constant. Find the conditions that the locus of P may be a circle. — Communicated by George B. Hicks, Cleveland, Ohio.

IV. If A and B represent the semi-axes of an ellipse, the altitudes of the minimum circumscribing isosceles triangles, having their vertices in the axes produced, are as $A : B$. — Communicated by James F. Roberson, Grantsburgh, Indiana.

V. Prove that, of all circular sectors having the same perimeter, the one of greatest area is that in which the circular arc is double the radius.

Solutions of these problems must be received by April 1st, 1861.

REPORT OF THE JUDGES UPON THE SOLUTIONS OF THE PRIZE PROBLEMS IN No. I., Vol. III.

THE first Prize is awarded to A. G. BARKER, Waterville College, Me.

The second prize is awarded to H. TEIMAN, Baltimore, Md.

The third prize is awarded to E. O. GIBSON, South New Berlin, N. Y.

PRIZE SOLUTION OF PROBLEM I.

By E. K. LEONARD, Marietta College, Ohio.

Let AB and CD be two diameters of a given circle, drawn at right angles to each other; AEB a circular arc described with radius DA or DB. Prove that the area of the lune $AEBC$ equals the area of the triangle ADB.

Let R be the radius of the arc AEB, and r that of the circle. When $ABC = \frac{1}{2}\pi r^2$, $ADBC = \frac{1}{4}\pi R^2$, $r : R = 1 : \sqrt{2}$. Substituting 1 and $\sqrt{2}$ for r and R, we have $ABC = ADBE$. Subtracting ABE from each, we get $AEBC = ADB$.

Most of the solutions of this problem are nearly the same as the above.

PRIZE SOLUTION OF PROBLEM II.

By E. O. GIBSON, South New Berlin, N. Y.

Of all isoperimetrical polygons having the same number of sides, the greatest is that which is equilateral.

Since an isosceles triangle has a greater area than any other triangle of equal base and perimeter, it follows that if any polygon, $ABCD$, &c., be a maximum among others of equal perimeters and the same number of sides, all the triangles, ABC, BCD, &c., must be isosceles, and have $AB = BC = CD =$ &c.

PRIZE SOLUTION OF PROBLEM III.

In any triangle, if P, P', and P'' denote the perpendiculars from the vertices to the opposite sides; p, p', and p'' the portions of the same between their point of intersection and the sides. Prove that

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

Mr. G. A. OSBORNE, Jr., says: Let a, b, c denote the sides of the triangle perpendicular respectively to P, P', P''. Then

$$a p + b p' + c p'' = 2 \text{ (area of triangle)},$$

and
$$a P = b P' = c P'' = 2 \text{ (area of triangle)}.$$

Dividing each term of the first equation by the term below it, gives

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

Mr. GIBSON notices that, in an obtuse-angled triangle, as the parts of the perpendiculars let fall on the sides produced, and which are represented by p and p', lie beyond those sides, they may be regarded as minus, and we still have

$$\frac{p}{P} + \frac{p'}{P'} + \frac{p''}{P''} = 1.$$

Mr. G. B. HICKS remarks, that this proposition may be included in the still more general statement, that, if from the angles of a triangle three lines, $A a$, $B b$, $C c$, be drawn to meet the opposite sides, and from any point, O, within the triangle three lines, $O \alpha$, $O \beta$, $O \gamma$, be drawn parallel to them to meet the sides, then

$$\frac{O \alpha}{A a} + \frac{O \beta}{B b} + \frac{O \gamma}{C c} = 1.$$

PRIZE SOLUTION OF PROBLEM IV.

By A. G. BARKER, Waterville College, Maine.

At what angle must the rudder of a vessel be inclined to the stream, that the effect produced may be a maximum?

Let the angle made by the rudder with the direction of the vessel be denoted by φ, the horizontal distance of the vessel's centre of gravity from the line of contact of the rudder by a, and the length of the rudder by b. The force exerted by the water upon the rudder parallel to the direction of the vessel's motion is proportional to $\sin \varphi$, since this force is zero for φ equal zero, and a maximum for φ

equal 90°. We may, therefore, take $\sin\varphi$ as its measure; and since φ is the angle this force makes with the rudder, its component perpendicular to the rudder is $\sin^2\varphi$, which we may suppose to act in the perpendicular passing through its middle point. This perpendicular produced will meet the line of direction of the vessel at a distance $\frac{\frac{1}{2}b}{\cos\varphi}$ from the line in which the rudder joins the vessel, and $a + \frac{\frac{1}{2}b}{\cos\varphi}$ is the length of the arm upon which the force $\sin^2\varphi$ acts at an angle $\frac{1}{2}\pi - \varphi$. The component of $\sin^2\varphi$ perpendicular to $a + \frac{\frac{1}{2}b}{\cos\varphi}$ is $\sin^2\varphi\cos\varphi$, and

$$u = \left(a + \frac{\frac{1}{2}b}{\cos\varphi}\right)\sin^2\varphi\cos\varphi = \text{max.}$$

$$\therefore \frac{du}{d\varphi} = 2a\sin\varphi\cos^2\varphi - a\sin^3\varphi + b\sin\varphi\cos\varphi = 0$$

$$= 3a\cos^2\varphi + b\cos\varphi - a = 0;$$

$$\therefore \cos\varphi = -\frac{b \pm \sqrt{(12a^2 + b^2)}}{6a}.$$

The second derivative gives

$$\frac{d^2u}{d\varphi^2} = -6a\cos\varphi\sin\varphi - b\sin\varphi,$$

which, for $\varphi < 180°$, is negative for the positive value of $\cos\varphi$; and therefore this value of $\cos\varphi$ will make u a maximum.

The negative value of $\cos\varphi$ gives

$$\frac{d^2u}{d\varphi^2} = \sqrt{(12a^2 + b^2)}\sin\varphi,$$

a positive quantity; and u is therefore a minimum for the negative value of $\cos\varphi$.

When $b = 0$, that is, when the length of the rudder is neglected, $\cos\varphi = \pm\sqrt{\frac{1}{3}}$; $\therefore \varphi = 54° 44'$, or $125° 16'$.

None of the solutions of Problem V. are thought worthy of publication.

Simon Newcomb,
W. P. G. Bartlett,
Truman Henry Safford.

DESCRIPTIVE GEOMETRY OF ONE PLANE.

CHAPTER I.

THE POINT, THE RIGHT LINE, AND THE PLANE.

The Point.

§ 1. A POINT in space is represented by its orthographic projection upon a fixed plane, called the *plane of reference*, and a number written near the projection, to show the distance of the point from this plane: this number is called the *reference* of the point with respect to the plane.

The plane of reference will at present be considered horizontal, and entirely below the magnitudes to be represented, thus avoiding any confusion which might arise from the use of negative references.

The Right Line.

§ 2. A right line in space is determined by the projections and corresponding references of two of its points.

§ 3. PROBLEM I. *Given the projection of a point situated upon a given right line, to find the reference of the point.*

Let AB (Fig. 2) be the projection of a line, and let the references of two of its points, projected at A and B, be respectively α and β. It is required to find the reference of the point, of which C is the projection. If we imagine the line in space revolved about its horizontal projection as an axis, so as to coincide with the plane of reference, it will take the position $A'B'$, making AA' equal to α, and BB' equal to β. Calling the unknown reference of C, γ, we have

$$BA : BC = \alpha - \beta : \gamma - \beta,$$

whence

$$\gamma = \frac{BC}{BA}(\alpha - \beta) + \beta.$$

The lengths of AB and BC are known, having been measured in units of the scale (Fig. 1), and α and β are measured upon the same scale.

Conversely, if it be required to find the projection of that point of the line whose reference is γ, we have, from the same proportion,

$$BC = AB\left(\frac{\gamma-\beta}{\alpha-\beta}\right).$$

Or the point C' may be found by laying off on the line AA' a distance AD equal to γ, and drawing DC' parallel to AB.

§ 4. Prob. II. *To find the distance between two given points.*

Let A and B (Fig. 2) be the given points, then evidently the distance required will be $\sqrt{(AB^2+(\alpha-\beta)^2)}$.

§ 5. *Definitions.* — The *tangent* of the angle made by the line with the plane of reference, $\left(\frac{\alpha-\beta}{AB}\right)$, is called the *declivity of the line*, and when put in the form $\frac{1}{m}$, it shows that points of the line whose projections are m units apart differ in reference by unity.

The distance between the projections of two points is called their *interval;* and the distance between the projections of two points of a line differing from each other by a unit of reference is called the *interval of the line.*

§ 6. By an application of the process explained in Problem I., it is easy to divide the projection of any given line into intervals; the projection thus divided is called the *scale of declivity of the given line.*

§ 7. Prob. III. *To construct the scale of declivity of a given line.*

Let the line be given by its projection, and the references 10.83 of the point A and 8.28 of B (Fig. 3); the interval of the points A and B, measured upon the scale (Fig. 1) is 5.75.

Therefore the declivity of the line is

$$\frac{10.83-8.28}{5.75}=\frac{2.55}{5.75}=\frac{1}{2.25}=\frac{1}{m}.$$

Therefore the interval of the line is $m = 2.25$. Therefore, setting off a distance 2.25 from A towards B, we have the point C, of which the reference is less by unity than that of A, i. e., 9.83.

But it is usual to divide the projection into integer references. To find the point 10, set off from A towards B a distance $2.25 \times 0.83 = 1.87$ of the scale (Fig. 1); from 10 set off a distance 2.25, which gives the point 9, &c. Divide one of the intervals, as 9 – 8, into 10 equal parts, and the scale will be completed.

§ 8. Prob. IV. *Through a given point to draw a line parallel to a given line.*

Let the line $a\,b$ be given by the projections and references of two points, 12 of a and 7.80 of b, and the point by its projection c and its reference 8.50. Draw through c a line parallel to $a\,b$, and lay off on it from c the interval $a\,b$; this gives a point d of the required line, of which the reference is $8.50 + 4.20 = 12.7$.

The Plane.

§ 9. The planes which are required to be represented are of two kinds, those which are limited in extent, as faces of polyhedrons, and those which are unlimited.

An unlimited plane is determined by the position of one of its lines of greatest declivity, that is, one of the lines of the plane making the maximum angle with the horizontal plane of reference; this line is evidently perpendicular to every horizontal line of the plane so defined.

The scale of declivity of this line is called the *scale of declivity of the plane*, and to distinguish it from that of an ordinary line it is made double.

A limited plane is represented by the projection of its perimeter, to which is added either its scale of declivity or the references of its angular points. Often a plane is given by two points of its line of greatest declivity.

§ 10. PROB. V. *To pass a plane through three given points.*

Let *a*, *b*, *c* (Fig. 5) be the points, of which the references are respectively 10.5, 7, and 4. Join *a* and *c*, and find on *a c* the point *d*, whose reference is the same as *b* (7); join *d* and *b*. This will be a horizontal line of the required plane; the line of maximum declivity will be perpendicular to it.

To find another point, draw through *c* a line parallel to *d b*; it will be the horizontal (4), which determines the line of greatest declivity.

If we have the projection of any point, as *e*, situated upon this plane, to find its reference, it will be sufficient to drop a perpendicular from *e* to the scale, and find by Prob. I. the reference of the point of intersection, which will be the reference required.

§ 11. PROB. VI. *To find the intersection of two planes.*

Let the two planes *M* and *N* (Fig. 6) be given by their scales of declivity.

In one plane, *M*, draw any two horizontals; through 7 and 11, for instance. In the other plane, *N*, draw horizontals having the same references, and the points *a* and *b*, their intersections, are two points of the line of intersection of the two planes.

It may happen that the horizontals of each plane are nearly or quite parallel. In the first case, take two auxiliary planes, *Q* and *P* (Fig. 7). *Q* intersects *M* in the line *x y*, and also *N* in the line *v t*; therefore *a*, the intersection of *x y* and *t v*, is a point common to the three planes *M*, *N*, and *Q*, or a point of the line of intersection of *M* and *N*. The same construction with *P* gives the point *b*; whence *a b* is the line of intersection required.

If the horizontals were exactly parallel, one auxiliary plane would be sufficient, since the line of intersection would be a horizontal.

§ 12. PROB. VII. *Through a given point in a given plane, to draw a line of the plane having a given declivity.*

Let *M* (Fig. 8) be the given plane, *a* the projection of the given

point, and 2.7 its reference; let the declivity of the required line be $\frac{6}{10}$. Draw any horizontal of *M*, as 6, and take the difference between its reference and that of the given point ($6 - 2.7 = 3.3$); multiply this difference by the reciprocal of the declivity of the required line ($3.3 \times \frac{10}{6} = \frac{11}{2} = 5.5$); then with the projection of the given point as a centre, and a radius 5.5, describe an arc *c d*; and where it meets the horizontal 6 will be another point of the required line.

There will be two solutions, if the declivity of the line be less than that of the plane; one, if it be equal, none, if it be greater.

§ 13. PROB. VIII. *Through a given line pass a plane of a given declivity.*

Let the declivity of the required plane be $\frac{1}{m}$ (Fig. 9). Conceive any point of the line, as *b*, to be the vertex of a cone, of which the elements have a declivity $\frac{1}{m}$; to trace the base of the cone, it will be sufficient to take a radius, *m*, and with *b* as a centre describe a circle; the plane of this base will be horizontal, whose reference is less by unity than that of *b*. Now, from the point *c* upon the given line, whose reference is less than *b* by unity, draw a tangent to the circle; this tangent will be a horizontal of the required plane. Draw a parallel through *b*, and we have another, whence its line of declivity is immediately determined. In general there will be two solutions.

Sometimes it happens, in the solution of this problem, that the line being very little inclined, the position of the second point would lie off the paper. The solution for this case is to take any other point, *d*, of the line, and consider it also the vertex of a cone having its base on the same horizontal plane, and its declivity the same, the radius of its base would be *m* times the altitude; a horizontal of the plane would then be found by drawing a tangent common to the two circles of the base. (Fig. 10.)

If the line given were exactly horizontal, then one cone would be sufficient.

§ 14. PROB. IX. *To find the point in which a line pierces a plane.*

Let M (Fig. 11) be the given plane, and A the given line. Through the line A pass any arbitrary plane, X, and find its intersection, $m\,n$, with M; the point in which this line of intersection meets A is the point sought.

This method may be used to ascertain the point in which two lines having the same projection intersect.

Pass through the two given lines any two arbitrary planes, X and Y, and determine their intersection; the point in which this intersection meets the given lines is their common point. Or the point of intersection may be found by revolving the lines about their common projection as an axis until they are parallel to the horizontal plane of reference. [The projection may be considered as situated in any horizontal plane, since, as in this case, it is often convenient to work upon horizontal planes of considerable references.]

Fig. 12 represents these two solutions.

§ 15. PROB. X. *From a given point in space to drop a perpendicular upon a given plane, to find the point in which it pierces the plane and the length of the perpendicular.*

First Solution. — The projection of the perpendicular will evidently be parallel to the scale of declivity of the plane, and its declivity the reciprocal of that of the plane.

Suppose the declivity of the plane is $\frac{1}{m}$, then the interval upon the perpendicular will be $\frac{1}{m}$.

Therefore, through the projection of the given point a draw a line parallel to the scale of declivity of the plane, and lay off from a on this line intervals of $\frac{1}{m}$, measured upon the scale of equal parts (Fig. 1); the references of the perpendicular will increase in a direction contrary to that of the scale of declivity of the plane.

The point in which the perpendicular pierces the given plane may be found by § 16, and its length by § 4.

If the line of declivity of the plane be given by the references of two of its points, c and d, the perpendicular can still be drawn without constructing the scale of the plane. For after drawing the parallel to the line of declivity of the plane, set off on it from the point a a distance equal to the difference of the references of c and d, and the reference of this last point will be the reference of a *plus* or *minus* the length of the interval between c and d, according as it is laid off *above* or *below* the point a. That is, intervals of the plane correspond to differences of reference on the perpendicular, and reciprocally.

There is another solution to this problem, which may be used to advantage in many of the subsequent problems.

Let a (Fig. 13) be the given point, de a section of the plane of reference, dc the line of maximum declivity of the given plane, ac the required perpendicular, and de its horizontal projection; $\tan cde = \frac{1}{m}$, the declivity of the given plane. Then the length of

$$ac = ad \cos \tan^{-1} \tfrac{1}{m}, \quad \text{and} \quad de = ad \sin \tan^{-1} \tfrac{1}{m} \cos \tan^{-1} \tfrac{1}{m};$$

these two may be readily constructed in connection with the scale of declivity of the plane.

Let M (Fig. 14) be the given plane, and a be the projection of the point of which the reference is 11. Let fall from a a perpendicular to the scale, and find the corresponding reference 8.4 of b. From b set off a distance bc equal to an interval of the scale M, and also set off bd equal to a unit of the scale of equal parts (Fig. 1); join c and d, and produce this line. From c set off a distance ce equal to 2.6, the difference of reference between b and a. Draw ef perpendicular to cd, and fh parallel to ce; and the point h, where the last line cuts the parallel through a to the scale M, is the point sought. ah is the

projection of the required perpendicular and cf is its length; the angle fce is the angle of declivity of the plane.

The interval ah may be easily calculated. Let the declivity of the plane be $\frac{1}{m}$, and the difference of reference between the points a and b be denoted by δ; then the length of the required perpendicular is

$$p = \delta \cos \tan^{-1} \frac{1}{m} = \delta \cdot \frac{m}{\sqrt{(m^2+1)}},$$

$$ah = \delta \cdot \cos \tan^{-1} \frac{1}{m} \sin \tan^{-1} \frac{1}{m} = \delta \frac{m}{\sqrt{(m^2+1)}} \cdot \frac{1}{\sqrt{(m^2+1)}} = \delta \frac{m}{m^2+1}.$$

The difference of reference between the point h and the horizontal ab is $\delta \sin^2 \tan^{-1} \frac{1}{m} = \delta \frac{1}{1+m^2}$.

§ 16. Prob. XI. *To draw through a given point a plane perpendicular to a given line.*

The projection of the scale of declivity of the plane must be parallel to that of the given line, and its declivity is the reciprocal of that of the line.

So the scale is found in the same manner as that of the required line by the first method of the last problem.

§ 17. Prob. XII. *To find the shortest distance from a point, a, in space to a line, A.*

First Method. — Pass through a a plane perpendicular to A, and find the point in which A pierces this plane; the line connecting this last point with a will be the perpendicular required.

Second Method. — Let a (Fig. 15) be the given point, and A the given line; join A and the point b of A having the same reference (7), ab will be a horizontal of their plane (X); rotate this plane about ab as an axis until it becomes parallel to the horizontal plane of reference; any point of A, as c, will be found at c', by taking sc' equal to the distance between the axis and c in space. Therefore A' will be the revolved position of A; ad' will be the length of the

perpendicular, and, making the counter revolution, we have $a\,d$, its projection.

§ 18. PROB. XIII. *Through any two lines, A and B, pass two parallel planes.*

Through any point of A draw a line parallel to B; these two lines will determine the first plane. Then through any point of B draw a plane parallel to the first.*

§ 19. PROB. XIV. *Through a given line, A, to pass a plane perpendicular to a given plane, M.*

From any point of A drop a perpendicular upon M; this perpendicular and the line A determine the required plane.

§ 20. PROB. XV. *To measure the angle of two lines, A and B.*

Draw through any point of A (Fig. 17) a line parallel to B, as $a\,y$, and pass a plane through A and $a\,y$. Rotate this plane about any horizontal $x\,y$, so as to be parallel to the plane of reference. The new positions of the lines determine the angle sought, $x\,a'\,y$.

Since the three sides of the triangle $x\,a\,y$ are very easily determined, the angle may be found without rotation by the formula,

$$\cos A = \frac{b^2 + c^2 - a^2}{2\,b\,c}.$$

§ 21. PROB. XVI. *Through a given point, a, to draw a line which shall make a given angle, μ, with a given line, A.*

Join a (Fig. 18) and the point of A having the same reference, this will be a horizontal of their common plane; rotate this plane till it is parallel to the plane of reference, and find A', the new position of the line A. Draw through a a line $a\,c'$, making the given angle μ with the line A', and make the counter revolution: $a\,c$ will be the line sought.

If the tangent of the angle μ be given, $\frac{1}{m}$, the point c' may be

* The construction for drawing one plane parallel to another is the same as for drawing one line parallel to another, and was explained by Prob. IV. Fig. 16 gives the construction.

found by dropping from a the perpendicular $a\,d'$, and measuring off the distance $d'\,c'$ equal to m times $a\,d'$.

§ 22. PROB. XVII. *To construct the angle of a given line, A, and a plane, M.*

From any point of A drop a perpendicular upon M, and by Prob. XV. find the angle between A and this perpendicular; this will be the complement of the angle sought.

§ 23. PROB. XVIII. *To find the scale of declivity of a line, knowing its projection, and also knowing it to be perpendicular to a given line, A, at a given point, a.*

Through the angular point a draw a plane, X, perpendicular to the given line. The unknown line will be in this plane; therefore drawing in X any horizontal, the point in which it meets the given projection of the unknown line will correspond to a point of the required line of the same reference; whence its scale may be immediately constructed.

§ 24. PROB. XIX. *To find the scale of declivity of a line, knowing its projection, and also the angle, μ, which it forms with a given line at a given point.*

If with the given angular point as a centre a sphere be conceived to be described, the four points of its surface in which the two lines and their two projections pierce it will form the angular points of a spherical quadrilateral, a, b, c, d (Fig. 19), in which $a\,b$ is the angle of declivity of the given line, $a\,d$ the given angle μ, $c\,b$ that of the projections, $a\,b\,c$, $b\,c\,d$ are right angles, and $d\,c$ is the unknown angle of declivity: whence the side $d\,c$ may be calculated by the rules of spherical trigonometry; but the graphic solution is as follows:—

Let a (Fig. 20) be the angular point, $a\,b$ the projection of the given line, $a\,e$ that of the unknown line. Pass a plane through the given line and the projection of the unknown line; its scale will be M, its angle of declivity $M\,f\,g$, the complement of which is $g\,f\,e$. Rotate

the line $a\,b$ about $a\,e$ as an axis, so as to coincide with the horizontal plane through a; it will take the position $a\,c$.

Make the angle $c\,a\,d$ equal to the given angle μ. Now, comparing Figs. 19 and 20, we have the three parts of the spherical triangle $a\,c\,d$ (Fig. 19), $a\,c$, $a\,c\,d$, and $a\,d$, which correspond respectively to $e\,a\,c$, $g\,f\,e$, and $c\,a\,d$ of (Fig. 20); and it is required to find the side $d\,c$ (Fig. 19), or, in other words, the other face of the pyramid. To do this, erect at any point m of $a\,e$ (Fig. 20) a perpendicular, $m\,n$; make the angle $m\,n\,r$ equal to $g\,f\,e$; then $m\,r$ is a line of the unknown face revolved about $n\,m$ as an axis, to coincide with the horizontal plane. At n erect a perpendicular to $m\,n$, and also draw $n\,v$ perpendicular to $a\,c$, and $n\,s$ perpendicular to $n\,v$ and equal to $n\,r$. The line $n\,v\,x$ is the horizontal trace of a vertical plane; and joining s and x, $s\,x$ is the revolved position of the intersection of the unknown face and the vertical plane $u\,x$. With v as a centre and $u\,v$ as radius describe an arc $u\,t$, cutting $s\,x$ in t. Let fall from t the perpendicular $t\,z$ to $u\,x$, and from z a perpendicular, $z\,y$, to $a\,e$. With x as a centre, and a radius $x\,t$, describe an arc meeting $z\,y$ produced in t'. Join t' and a, and we have $e\,a\,h$, the angle of declivity sought, and $a\,h$, the position of the line required revolved about its horizontal projection.

§ 25. Prob. XX. *To reduce an angle to the horizon; that is, having given μ the angle which a line forms with a given line at a certain point, and also its declivity, to find its projection.*

Let $a\,b$ (Fig. 21) be the given line, and a the angular point. Revolve the line about its projection so as to coincide with the horizontal plane through b; it will take the position $b\,a'$. Make the angle $a\,a'\,c$ equal to the angle of declivity of the unknown line, and $b\,a'\,d$ equal to μ, the given angle.

$a'\,e$ intersects $a\,b$ in c, therefore with a as a centre, and a radius $a\,c$, describe a circle; it will be the base of a cone of which a is the vertex, and every element of this cone will have the given declivity.

Take $a' d$ equal to $a' c$, and with b as a centre and $b d$ as a radius describe an arc intersecting the former in the points m and n; $a m$ or $a n$ will be the projection required. The line required and the given line are represented in revolved position by $b a' d$, and the third side of the triangle $b a' d$ is horizontal.

§ 26. *Transformation of the Plane of Reference.* — It is convenient often to change the plane of reference and refer the magnitudes to another plane whose scale with respect to the primitive is given; this may be done by an application of Prob. X.

I. *To change with respect to a point*, i. e. *to find the new projection and reference of a given point, a, with respect to any plane, M.*

Let fall from a a perpendicular upon M; find the point a', in which it pierces the plane M, and the length of the perpendicular.

The point a' is the new projection, and the length of the perpendicular is the reference sought.

II. *To change with respect to a line.*

Find the new projections and references of two points of the line. The construction may be simplified by taking as one point that in which the line pierces the plane M, which is its own projection, and has a reference zero.

III. *To change with respect to a plane; change with respect to any three points of the plane.*

The construction is much simplified by finding first the line of intersection of the given plane and the new plane of reference; this line is the new horizontal of zero;* dropping upon this intersection a perpendicular from the new projection of another point of the plane, this will be the projection of the new scale, which is known by the references of two points.

The scale of the primitive with respect to the new plane of refer-

* The term horizontal here is evidently a misnomer, but it cannot lead to any ambiguity.

ence will have the same projection and interval, but will be the negative of it, *i. e.* it will increase from zero in the opposite direction.

The position of the new plane of projection may not be given directly, it may be required to be parallel or perpendicular to a given line or plane. The Problems IV., X., and XI. enable us to find immediately the scale of declivity required.

§ 27. PROB. XXI. *To find the angle of two planes, M and N.*

First Method. — Take the plane M as a new plane of reference; the new angle of declivity of N will be the angle sought. Fig. 22 gives the construction. Find the intersection $c\,e$ of the two planes M and N; this is the new horizontal. Let fall from any point of N, as a, a perpendicular upon M; it pierces it at a', and its length is $b\,d$. Let fall from a' a perpendicular, $a'f$, to $c\,e$, the new horizontal; then $a'f$ is the line of declivity of N with respect to M. Erect at a' a perpendicular to $a'f$ equal to $b\,d$, the new reference of a'; join g and f; $f\,g$ is the revolved position of the line of maximum declivity of N', and $a'f\,g$ is the angle required.

Second Method. — Let fall from any point in space, a, a perpendicular upon each plane, and measure the angle of these perpendiculars by § 20; this will be the supplement of the angle required. Or the length of the perpendicular upon either plane, as M, divided by the distance from its foot to the point in which the perpendicular upon N pierces the plane M, will be the tangent of the angle sought.

Third Method. — Let M and N (Fig. 23) be the two planes, $a\,b$ their line of intersection. Conceive a plane drawn perpendicular to $a\,b$ at the point y; $x\,y\,z$ will be a horizontal of this plane, having a reference 70. From y let fall a perpendicular, $y\,y'$, upon $a\,b'$, the revolved position of the line of intersection of the planes, and take $y\,y''$ equal to $y\,y'$, the angle $x\,y''\,z$ will be the angle sought.

§ 28. PROB. XXII. *Through a given right line to pass a plane which shall make a given angle, μ, with a given plane.*

Let A (Fig. 24) be the given line, and M the given plane. Take M as the new plane of reference.

To find the new position, A', of A, find first the point a, in which A pierces M; and then from any point of A, as d, let fall a perpendicular upon the plane M; it pierces this plane at the point c, and its length $e\,x$ is the new reference of c; therefore $a\,c$ is the new projection A'. Denote tan μ by $\frac{1}{m}$, the given declivity. Conceive any point of A', as k, the vertex of a cone, whose elements have the given declivity. The base of this cone is a circle situated in the new plane of reference, having for a centre k, and for a radius the reference of k multiplied by m.

Drawing from a a tangent to this circle, it will be a horizontal of the required plane with respect to the new plane of reference. This new horizontal meets the horizontal of zero of the primitive plane at the point y, therefore y is common to the primitive plane, to the new plane, and to the required plane; it is also its own projection in both planes, and has a reference zero. Therefore, joining y and the point z of the line A, whose reference is zero, we have the line $z\,y$, a horizontal of the required plane with respect to the primitive plane. Any other parallel, as through 40, is sufficient to determine the scale of declivity N of the required plane. The problem has in general two solutions. The other plane is determined by the other tangent, $a\,y'$, and has for a scale N'.

Second Method. — Let $\frac{1}{m}$ be the tangent of the given angle, A the given line, and M the given plane. From any point of A let fall a perpendicular upon M, and measure its length, p. Measure also the distance from the foot of p to the point in which A pierces M, and call this distance A'.

The distance from the foot of p to the intersection of the unknown plane with m is $m \cdot p$, and the sine of the angle which A' makes with the same intersection is $\frac{m \cdot p}{A'}$.

Through the foot of p pass a plane perpendicular to A'; and upon its intersection with M, starting from the foot of p, lay off a distance equal to $A' \times \tan\left(\sin^{-1}\frac{p \cdot m}{A'}\right)$, or $\frac{A \cdot p \cdot m}{\sqrt{(A'^2 - p^2 \cdot m^2)}}$; and this point will be one of the required planes which, with the given line, immediately determines it.

§ 29. PROB. XXIII. *To determine the perpendicular between two right lines not in the same plane.*

First Solution. — Let A and B (Fig. 25) be the two lines. Pass a plane, X, perpendicular to A at any point, c, and take it as the new plane of reference. The new projection of B is B', and that of A is reduced to the point c. Draw cr perpendicular to B'; this will be the perpendicular required. Since cr in space is parallel to the plane X, the new reference of c (the point in which cr meets the new projection of A) is the same as that of r. Therefore, returning to the primitive plane of reference, the point r is projected in t, and making cs in space equal to the new reference of r, st will be the projection of the perpendicular between the lines A and B, and cr will be its length.

Second Direct Solution. — Let A and B (Fig. 26) be the given lines. Pass through B a plane, X, parallel to A. Then from any point of A, as a, let fall a perpendicular to X; it pierces the plane X in a', and cd is its length. A line through a parallel to A will be the projection of A upon the plane X. The point n, in which this projection meets B, is one point of the perpendicular sought; drawing mn parallel to aa', we have the required projection of the perpendicular distance between A and B.

§ 30. PROB. XXIV. *Given a plane, M, and the projection of a limited line, A, of this plane to construct upon A a cube which shall rest upon M.*

Let ad (Fig. 27) be the length, A, of the given projection. Make the angle sxy equal to the angle of declivity of M. Rotate the plane M about any horizontal x, as 50. To find the new position of

d, let fall from d a perpendicular to $s\,x$, and produce it to n on the line $x\,y$; then $x\,n$ will be the distance of d from the axis of rotation, and will make known the side $a\,d'$ of the cube. The square $a\,d'\,c'\,b'$ will be the base in revolved position, making the counter rotation, the base will be projected in $a\,b\,c\,d$. To find the upper base after revolution, make $x\,y$ equal $a\,d'$, and let fall the perpendicular $y\,s$; this will be the length of the horizontal projections of the edges of the cube projected at $a\,b\,c\,d$; whence, drawing through a, b, c, d lines perpendicular to the axis of rotation and equal to $y\,s$, we have the upper vertices; and the difference of reference between any two points symmetrically situated, as b and h, is $s\,x$, whence the cube is completely determined.

§ 31. Prob. XXV. *To find the intersection of two polyhedrons.*

Solution. — Pass a number of horizontal planes through each solid. Then the intersections of horizontals of the same reference, drawn in the intersecting adjacent faces of the two polyhedrons, make known the lines of intersection of those faces.

Fig. 28 represents an application of this method to the determinations of the intersections of two quadrangular pyramids.

§ 32. Prob. XXVI. *To find the shadow of any polyhedron.*

Let $a\,b\,c\,d\,e$ (Fig 29) be the lower base, and $a'\,b'\,c'\,d'\,e'$ the upper base of a polyhedron. Let the reference of the upper base be 80, and that of the lower base 0. Instead of supposing, as usual, the rays of light parallel to the diagonal of a cube resting upon the horizontal plane, let them be supposed to make an angle of 45° with the plane of reference, and have a direction indicated by the arrow A.

To find the shadow of any point, as f', draw through f' a parallel to A, and lay off a distance $f'\,f''$ equal to the reference 80 of f'. The same constructions for d' and e' give the points d'' and e''. Joining d and d'', f and f'', we have $f\,f''\,e''\,d''\,d$ as the shadow required.

NOTES AND QUERIES.

I. *A General Method of finding Tangents to Algebraic Curves.* — If the curve be referred to rectangular axes, and $x_1 y_1$ be the co-ordinates of the point of tangency, the equation of a line passing through this point will be of the form,

$$y - y_1 = a(x - x_1).$$

It is required to find the value of a when this line is tangent to the curve. Refer the curve to a system of polar co-ordinates, whose pole is at the point $x_1 y_1$, and axis parallel to the axis of x. Since the pole is upon the curve, the absolute terms will disappear from this equation. Every term of the equation will now contain r, the first power of which may be divided out. Then making $r = 0$, find the corresponding value of $\tan \theta$, which will evidently be the required value of a. This value of $\tan \theta$ may always be found by the solution of an equation of the first degree. For any term that contains the product of $\sin \theta$ and $\cos \theta$, or the higher powers of either, will also contain the higher powers of r, and will therefore disappear when $r = 0$. Hence it readily appears that in practice we may neglect all terms except those which contain the first power of r; and from these we shall find the required value of $\tan \theta$.

This method will readily suggest its own demonstration.

Examples.

1. Take the folio of Descartes, of which the equation is

$$x^3 - 3cxy + y^3 = 0.$$

The equations of transformation are

$$x = x_1 + r \cos \theta, \qquad y = y_1 + r \sin \theta.$$

Hence we have

$$\left.\begin{array}{l} + x_1^3 + 3\,x_1^2\,r\cos\theta + \text{terms involving } r^2, \text{ \&c.} \\ + y_1^3 + 3\,y_1^2\,r\sin\theta + \quad " \qquad " \qquad " \\ - 3\,c(x_1 y_1 + x_1\,r\sin\theta + y_1\,r\cos\theta + \text{terms involving } r^2, \text{\&c.} \end{array}\right\} = 0.$$

Striking out the terms independent of r, since

$$x_1^3 - 3\,c\,x_1 y_1 + y_1^3 = 0,$$

dividing by r, and then making $r = 0$, we have

$$3\,x_1^2\cos\theta + 3\,y_1^2\sin\theta - 3\,c\,x_1\sin\theta - 3\,c\,y_1\cos\theta = 0.$$

$$\therefore \tan\theta = \frac{c\,y_1 - x_1^2}{y_1^2 - c\,x_1};$$

and this result agrees with that obtained by differentiation.

2. The general equation of all parabolas is $y^m = a^{m-1}\,x$.

The transformed equation is

$$y_1^m + m\,y_1^{m-1}\,r\sin\theta + \text{\&c.} = a^{m-1}\,x_1 + a^{m-1}\,r\cos\theta.$$

Therefore, by neglecting the terms independent of r, dividing by r, and then making $r = 0$, we get

$$m\,y_1^{m-1}\sin\theta = a^{m-1}\cos 0.$$

$$\therefore \tan\theta = \frac{a^{m-1}}{m\,y_1^{m-1}},$$

the true result.

This method may be extended to transcendental curves by means of algebraic expansion. — Walter Holliday, University of Virginia.

II. *Squares of Numbers ending in* 5, 25, *and* 75. — If in the square of the binomial $(t+u)^2 = (t^2 + 2\,t\,u + u^2)$, we put u equal to 5, or one half ten, it reduces to $(t^2 + t + \frac{1}{4}) = t(t+1) + \frac{1}{4}$; but since the square of a number ending in 5 always ends in 25, we derive the following rule for squaring numbers ending with 5.

Square the unit's figure, annex to the left of its square the product of the ten's figure multiplied by itself plus unity.

Thus, $(25)^2 = 625$, $(65)^2 = 4225$, $(95)^2 = 9025$.

For numbers ending in 25 and 75 let h represent the hundreds, then 25 and 75 will be represented by $\frac{1}{4}$ and $\frac{3}{4}$; annexing each of these fractions to h, we have the two binomials $(h+\frac{1}{4})$ and $(h+\frac{3}{4})$; squaring each and factoring, they become

$$[h(h+5)+\tfrac{1}{16}], \qquad [(h+1)(h+5)+\tfrac{1}{16}],$$

each of which may be readily expressed in the form of a rule, by observing that the square of numbers ending in 25 or 75 always ends in 625. Thus,

$(225)^2 = 50625$, $(725)^2 = 525625$, $(375)^2 = 140625$, $(975)^2 = 950625$.

Let $h = 100$ and $t = 10$, then we have, for the powers of the above numbers,

$$(225)^2 = (\tfrac{9}{4}h)^2 = \tfrac{81}{16}h^2 = 50625,$$
$$(375)^2 = (\tfrac{15}{4}h)^2 = \tfrac{225}{16}h^2 = 140625,$$
$$(225)^3 = (\tfrac{9}{4}h)^3 = \tfrac{729}{64}h^3 = \tfrac{729}{8\times 8}h^3 = \tfrac{91125}{8} = 11390625,$$
$$(25)^4 = (\tfrac{5}{2}t)^4 = \tfrac{625}{16}t^4 = 390625,$$
$$(15)^6 = (\tfrac{3}{2}t)^6 = \tfrac{729}{64}t^6 = \tfrac{729}{8\times 8}t^6 = \tfrac{91125}{8} = 11390625.$$

Numbers may also be multiplied together in the same manner. Thus,

$$(15\times 25\times 5) = (\tfrac{3}{2}t\times\tfrac{1}{4}h\times\tfrac{1}{2}t) = \tfrac{3}{16}t^2h = \tfrac{3}{16}h^2 = 1875,$$

$$(35\times 75\times 65\times 5) = (\tfrac{7}{2}t\times\tfrac{3}{4}h\times\tfrac{13}{2}t\times\tfrac{1}{2}t) = \tfrac{273}{8\times 4}t^3h = \tfrac{34125}{4} = 853125.$$

— James F. Roberson, State University, Bloomington, Indiana.

III. *Solution of Problem V., No. X., Vol. II.* — Suppose the sphere to be divided into an infinite number of concentric spherical shells; let A be their common centre, and B the exterior point; and M any particle on the surface of one of these shells, $r = AM$, $c = AB$, $y = BM$, $dr =$ the thickness of the shell, and $\rho =$ its density.

Since dr is infinitely small, ρ may be considered constant for the

same shell, but varying for *different* shells. The resultant attraction of the whole shell on

$$B = \frac{\pi \varrho r\, dr}{c^2} \int_{c-r}^{c+r} \left(1 + \frac{c^2 - r^2}{y^2}\right) dy = \frac{4\pi \varrho r^2\, dr}{c^2}.$$

To obtain the part of the shell which exerts half this attraction, take $c - r$ and x for the limits of y, and put

$$(1) \qquad \frac{\pi \varrho r\, dr}{c^2} \int_{c-r}^{x} \left(1 + \frac{c^2 - r^2}{y^2}\right) dy = \frac{2\pi \varrho r^2\, dr}{c^2}.$$

But $\int_{c-r}^{x} \left(1 + \frac{c^2 - r^2}{y^2}\right) dy = x - \frac{c^2 - r^2}{x} + 2r$; therefore equation (1) gives $x - \frac{c^2 - r^2}{x} + 2r = 2r$, or

$$(2) \qquad x = \sqrt{(c^2 - r^2)}.$$

If C be on the surface of the shell, and $BC = x$, the triangle ABC will be right angled at C. If we suppose r in equation (2) to take successively all values between zero and the radius of the sphere, the triangle ABC will be right angled in every case; hence the locus of the point C, which is always on the bisecting surface, is the surface of the sphere described on AB as diameter. — Asher B. Evans, Madison University, Hamilton, N. Y.

IV. *On an Approximate and Graphical Rectification of the Circle.** — The following treatment of this oft-attempted problem is founded upon the singularly close approach that is made by the angle $\tan^{-1}(\frac{1}{4}\pi)$ to a root of the equation $\sec x = \cot x$. The tabular solution of this equation is presented in the form

$$\cos x = \tan x = \sqrt{\left(\frac{\sqrt{(5)} - 1}{2}\right)} = 0.7863,$$

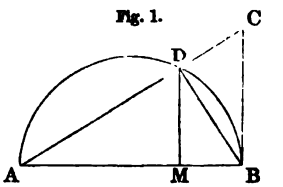
while $\frac{1}{4}\pi = 0.7854$; so that $x = 38°\,10'\,46''$, and $\tan^{-1}\frac{1}{4}\pi = 38°\,8'\,46''$:

* From the London Quarterly Journal of Pure and Applied Mathematics, October, 1860.

from which it is clear that if a simple construction can be found for obtaining the angle x that satisfies this equation, the tangent of that angle will yield a very close approximation to the desired periphery of a circle.

Let the chord $A\,D\,C$ (Fig. 1) cut the semicircle $A\,D\,B$ in D, and the tangent $B\,C$ in C, in such a manner that $A\,D = B\,C$; then shall $A\,D$, $B\,C$ represent *quam proximé* quadrants upon the semicircle $A\,D\,B$. By similar triangles, $\frac{A\,D}{A\,M} = \frac{A\,B}{A\,D}$, $= \frac{A\,B}{B\,C}$, by hypothesis. Or sec $D\,A\,B = \cot D\,A\,B$, and $D\,A\,B$ is such an angle as x; therefore

Fig. 1.



$$B\,C = A\,B \tan D\,A\,B = 2 \tan x, \text{ if the radius be unity;}$$
$$= 1.5727; \text{ and } \tfrac{1}{2}\pi = 1.5708.$$

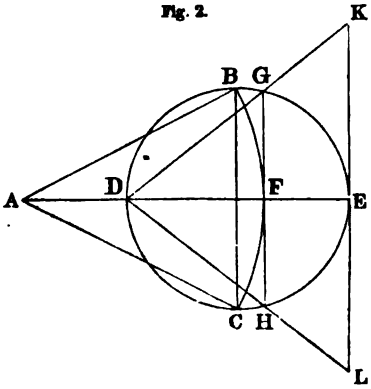
Therefore $B\,C$ exceeds the quadrant of $A\,D\,B$ by one thousandth part nearly; and in consequence, the triangle $A\,B\,C$ exceeds the semicircle $A\,D\,B$ by one thousandth part. It is upon such a singular coincidence as this that the following simple rule for approximate rectification of the circle with compass and ruler has been framed.

Rule.—"From an external point, at a distance equal to the radius from the given circle, describe a circle cutting the given circle at the two extremities of that diameter which is perpendicular to the line joining the point with the centre of the circle; and perpendicular to the same line draw two tangents to the two circles, the first touching the original circle at the point most remote from the external point, and the second touching the other circle, and cutting the original circle in two places. Then the two chords joining these two last points with the point of the given circle nearest to the external point, will represent the two quadrants that are there terminated; and if these chords be produced until they meet the tangent first

drawn, they will intercept upon it a portion bisected at the point of contact, and representing the two quadrants that are there terminated: and the triangle which this tangent forms with the two chords so produced will represent the area of the circle: and each of these rectilinear quantities will exceed the corresponding curvilinear quantity which it approximately represents by one thousandth part of the whole, nearly."

In proof of this construction, it will be seen that both DG and EK (Fig. 2) have by the construction the numerical values $\sqrt{[2\{\sqrt{(5)}-1\}]}$, radius being unity, and are therefore equal to one another, whence the inference proceeds as before.

Fig. 2.



This construction closely rivals in descriptive geometry the well-known arithmetical approximation to the value of π that is found in the vulgar fraction $\frac{22}{7}$, this fraction exceeding the true value by only four, and the above geometrical result exceeding the same by only twelve parts in ten thousand. This simple process will, for example, yield for the quadrant of a circle ten inches in diameter, a result exceeding the truth by less than the hundredth part of an inch, which amply suffices in accuracy for all that such processes can have in view.

P. S. — It has been pointed out to me that the equation $\sec x = \cot x$, which the angle $\tan^{-1}\frac{1}{4}\pi$ so nearly solves, is equivalent to that proposition which Herodotus has recorded concerning the dimensions of the great pyramid of Gizeh, when he says that the area of the sloping side is equal to the square upon the perpendicular height. The semiangle contained at the vertex between two opposite sloping sides is in this case such an angle as x, and

accordingly the above construction may be considered as affording a ready means for reproducing upon paper those proportions which a recent writer* has affirmed that this monument was intended to perpetuate, the periphery of its base bearing to its height very nearly the proportion of the circumference of a circle to its radius. The triangle AKL in the above construction will accordingly represent the vertical section of the Great Pyramid parallel to its sides. — A. S. HERSCHEL, Collingwood, England, August 18, 1860.

Mathematical Monthly Notices.

General Problems from the Orthographic Projections of Descriptive Geometry; with their Applications to Oblique — including Isometrical — Projections, Graphical Constructions in Spherical Trigonometry, Topographical Projection ("One Plane Descriptive"), and Graphic Transformations. By S. EDWARD WARREN, C. E., Professor of Descriptive Geometry and Geometrical Drawing in the Rensselaer Polytechnic Institute, Troy, N. Y. New York: John Wiley, 56 Walker Street, 1860. 8vo. pp. xxxv, 412.

The aims of PROFESSOR WARREN in this work are thus stated in his Preface: —

"1st. To proceed upon the basis of a perfectly accurate and rational idea of what Descriptive Geometry is.

"2d. To make the course perfectly *determinate* and rational, even to the very details, in the plan of its composition and the order of its progression.

"3d. To embody in its methods of treatment, as well as in its preparatory directions to the student, safeguards against false modes of study and sham scholarship.

"4th. To make the course sufficiently full to meet the wants of advanced students, and of the members of the professions based on mathematics and the physico-mathematical sciences."

The plan of the course is founded on the division of Surfaces into Ruled Surfaces and Double-curved Surfaces, and the subdivision of Ruled Surfaces into Plane Surfaces, Single-curved Surfaces, and Warped Surfaces. Upon each of these four classes of surfaces four operations are distinguished, namely, Projections, Intersections, Tangencies, and Developments.

In carrying out this plan the *General Problems* are placed under the two heads of *Lower Descriptive Geometry* and *Higher Descriptive Geometry*. Under the first head are treated quite fully the various problems relating to the plane, straight line, and point; those relating to the cone and cylinder, including conic sections; those relating to the hyperboloid of one nappe, and the hyperbolic paraboloid; and lastly, those relating to the sphere, ellipsoid, elliptic paraboloid, and hyperboloid of two nappes. Under the second head we find the construction of various spirals, the cissoid, conchoid, involutes, cycloids, helix, and spherical epicycloid; also problems on the developable helicoid, warped oblique arch, oblique helicoid, right conoid, and annular torus.

* "The Great Pyramid, and why it was built," by JOHN TAYLOR.

Book II. gives the *Applications of Orthographic Projections*. These include Isometrical Projections, Conventional Oblique Projections, Spherical Trigonometry, Topographical Projections (a name which the author prefers to One-plane Descriptive Geometry), and Graphic Transformations, which include changing the planes of projection and rotating given magnitudes. A collection of Miscellaneous Problems, sixty-three in number, closes the work.

From the above it will be seen that PROFESSOR WARREN'S book is far more comprehensive than any work on Descriptive Geometry before offered to American, or, so far as we know, to English readers. As such it will doubtless be welcomed by the Professors in our Scientific Schools, and by many students to whom the extensive French works on this subject are inaccessible. But, besides its comprehensiveness, this book has many positive merits. The author appears to have studied his subject well, and to be master of its minutest details. He is without doubt familiar with the works of his predecessors; but the marks of independent thought are apparent on even a cursory examination. His experience as a teacher has enabled him to judge of the wants of students, and to make many useful suggestions in the shape of remarks. We find also several examples of extended discussions of problems, that show a praiseworthy love of thoroughness.

With this general recognition of the merits of the book before us, we pass to some more particular remarks upon a few points.

The true sphere of Descriptive Geometry is one of the topics discussed in the Preface, and the conclusion arrived at is, that Descriptive Geometry is "the *exact graphic art*, and not the science of geometry, and is distinctively, *graphically operative geometry*, or the *Geometry of Graphic Problems*." In this the author agrees with CHASLES (LIOUVILLE'S *Journal*, 1847, p. 33), who also denies to Descriptive Geometry the name of a science. But there are many able supporters of the opposite view, among whom may be noticed MONGE, CARNOT, PONCELET, and OLIVIER. M. OLIVIER discusses the subject quite fully in the preface to his *Additions au Cours de Géométrie Descriptive*, and becomes quite indignant against those who deny the claims of Descriptive Geometry to be called a science as well as an art.

PROFESSOR WARREN has also some remarks on the use of models and pictorial diagrams in teaching Descriptive Geometry, and condemns their general use, on the ground that without them certain powers of the mind — attention, abstraction, and especially conception — are better cultivated. This objection would be more forcible, if these powers could be cultivated in no other way, or if there were not many persons to whom Descriptive Geometry would be very useful, if it could be brought within their grasp without long and to them painful study. The truth is that no real simplification of any science is to be dreaded, even when we regard study as a discipline of the mind. When fields once inaccessible to ordinary minds are annexed to the common domain, there will always be higher fields opening up to exercise and strengthen the loftiest faculties.

OLIVIER, with an experience of more than twenty years in teaching Descriptive Geometry, recommends the use of models in the strongest terms. The costly models bearing his name are well known; but he has also contrived a simple apparatus which any one can have made for himself, and which we shall here describe for the benefit of those who have not access to this author's works. Take two shallow boxes similar to the tables of a backgammon board, and connected like them by hinges. Fill up these boxes with sheets of cork, and attach a brace to hold one table at right angles to the other, when necessary. These are the planes of projection. Provide four sets of rods from one eighth to one fifth of an inch square, and of various lengths from four inches to twelve, or sixteen, or twenty four inches, or even longer, according to the size of the planes. These rods are to be pointed with needles, that they may be readily fixed in any position in the cork. They are to be painted, one set red, one red and white in

bands, one black, and one black and white in bands. The red rods are used to construct in space any system referred in position to the two planes of projection; the red and white rods to project the points of this system on each of the planes of projection; the black and white to project these projections on the ground line; and the black to represent the projections of the red lines in space. When by means of these rods the operations necessary for the solution of a problem have been completed, the red and the red-and-white rods are removed, and the vertical plane revolved on its hinges to a horizontal position. The work is then done. By this apparatus also all ruled surfaces, such as cones, cylinders, hyperboloids of one nappe, hyperbolic paraboloids, conoids, &c., may be exhibited.

The classification of subjects adopted by PROFESSOR WARREN will arrest the attention of every one. Few, we think, will approve it. System is in itself a great good; but here the dividing, subdividing, and dividing again, are carried so far, that one almost doubts whether the confusion incident to so many heads is not greater than that of no classification whatever. A few distinct heads, such as those generally adopted by writers on this subject, are all that are necessary, at least in a text-book.

The style is also frequently open to objection. The greatest simplicity, directness, and precision should characterize such a work. All long and wordy sentences, all parade and circumlocution, all vagueness and ambiguity, are serious hindrances to the student. As an example of prolixity (one of the worst perhaps), we make an extract from the chapter on Graphic Transformations.

"426. Beneath all the formal problems of Descriptive Geometry, and giving character to their mode of solution, is a tacitly recognized problem, founded in the failure of human inventive skill to keep pace with human knowledge, as seen in the incapacity, hitherto apparent, to make *available*, that is, *cheap* and *permanently reliable*, instruments for the *easy* and *exact* graphical construction, by a continuous movement, of any lines but right lines and circles, — other lines being usually determined by points which are afterward joined by hand, or by the use of the 'irregular curve' (ruler).

"427. In continuation of this reasoning, it appears: 1st. That any point revolves about a fixed axis in a circular arc; 2d. In order to secure an adequate, that is, an immediately available, representation of such an arc, it must appear circular or rectilinear in projection; 3d. For this purpose, the plane of the arc must be parallel or perpendicular to the plane of projection; and 4th. This being the case, the axis of the arc will be respectively perpendicular or parallel to the same plane of projection."

We understand by this simply, that, since straight lines and circles are more easily drawn than other lines, it is best to choose an axis of rotation either parallel or perpendicular to the plane of projection; because then the path of a rotating point will be in projection a straight line, or an arc of a circle.

On page 5 we find: "Any magnitude, except a point, which is irreducible to anything more simple, may be regarded as the result of the motion of some simpler magnitude, according to a certain law." Here it is twice implied that a point is a magnitude; while on page 1 it is said that, "Magnitudes are definite portions of *space*, having one, two, or three dimensions." A point is therefore a definite *portion of space*, and has at least one dimension, — probably three, since its length, breadth, and thickness are very likely equal.

On page 104 we find: — "All sections of an oblique cone with a circular base which are parallel to the base, are circles, whose diameter depends on their distance from the vertex. All other sections, whose planes cut all the elements, are ellipses, save the sub-contrary circular section, *EF*, Pl. VIII., Fig. 53, whose plane makes an angle with an element, as *VB*, equal to the angle made by the parallel plane, *CD*, with the opposite element, *VA*."

Here are two omissions. It should have been stated, that the angles with *VB* and *VA* are both to be taken on the side towards the vertex. Moreover, there is no intimation that the figure does not represent *any* section of the cone through opposite elements; while it is essential that it should represent the particular section made by a plane drawn through the axis perpendicular to the plane of the base.

In drawing tangents to curves, the author appears to have had in view the method of ROBERVAL. This method is thus stated by its author (*Mém. Acad. des Sciences*, VI. p. 22): —

"*Axiom.* The direction of the motion of a point which describes a curve is a tangent of the curve in each position of this point.

"*General Rule.* By the specific properties of the curve (which are given) examine the different motions which the describing point has at the place where you wish to draw the tangent: compound these motions into one, and draw the line of direction of the compound motion; you will have the tangent of the curve."

PROFESSOR WARREN says (p. 72): —

"The tangent to a curve is therefore the direction in which the generatrix, when at the point of tangency, would move were it to cease to generate a curve, and actually to move in the direction of its tendency for the instant.

"But the law of the generation of a curve is such that the tendency of the generatrix at any instant is the resultant of two component tendencies, while from any common school philosophy we know that such a resultant corresponds to the diagonal of a parallelogram, whose two adjacent sides express the relative values and directions of the component tendencies."

Then follow applications to the conic sections. On pages 284, 385 (the subject being higher curves) we find the method restated thus: —

"The tangent line to the curve, at any point, is the resultant direction of the motion of the generatrix, due to the two component motions of that generatrix.

. . . . .

"The usual case is that neither of these motions is uniform. In this case, the tangent at any point is the resultant of the two tendencies for that instant only. Hence to construct the tangent at any point, draw two straight lines through that point in the directions in which the generatrix tends to move, at that point, and lay off on these lines distances whose ratio shall equal the *ratio* of the *rates* at which the generatrix tends to move at the instant in the given directions. The diagonal of the parallelogram, formed on these distances as two of its sides, will be the required tangent.

"To furnish such momentary ratios as are necessary to the above solutions is one object of the higher calculus."

Now the principle of ROBERVAL'S method is acknowledged to be correct; but in applying it there are undoubtedly difficulties in determining what the components are. ROBERVAL himself, MONTUCLA, and even MONGE, in one case, are said by DUHAMEL (*Mém. Sav. Étrang.*, V. 1838, p. 257) to have confounded the component velocities of the generatrix with the projections of the total velocity upon the directions of the components. He shows also, that this confusion will produce no actual error, when the velocities are equal, or when the directions of the motions are at right angles to each other. But when a curve is given by an equation between the distances of its points to two or more foci, these conditions do not in general exist, and the error becomes manifest. To illustrate, suppose we had the equation $ar + br_1 = a$ *constant*, r and r_1 being the distances from any point of the curve to two fixed points, and a and b any multipliers, and let ϵ and ϵ_1 be the angles made by the tangent to the curve at any point with r and r_1. Then it is shown by DUHAMEL and by SERRET (*Méthodes en Géométrie*, p. 53, *et seq.*) that

$$\frac{dr}{dr_1} = -\frac{b}{a} = \frac{\cos\varepsilon}{\cos\varepsilon_1};$$

and, consequently, as will be readily seen by drawing a figure, that if we lay off on r and r_1 any distances having the ratio $dr : dr_1$, and erect perpendiculars at the extremities of these distances, the intersection of the perpendiculars will give the direction of the required tangent. It is obvious that the direction will not be that of the diagonal of a parallelogram constructed on the distances laid off as above, unless either $dr = \pm dr_1$, or the angle between r and r_1 is a right angle.

When, therefore, this method is used, the mode of finding the ratio of the components should be distinctly laid down. We have seen that it is possible to fall into error, even when the Differential Calculus is relied on to furnish the ratios. At any rate, if we have to rely on the Calculus for this essential step, the method ceases to be an independent one, and we had better adopt at once the recognized methods of drawing tangents derived from Analytical Geometry and the Calculus.

Extended as these remarks have become, we have left unnoticed many points that deserve attention. Enough, however, has been said to call attention to the principal features of this work. While we have not hesitated to point out what we consider faults, we trust that we properly appreciate the labors of the author in giving to the English student so extended, and, in many respects, so useful a work.

The Lady's and Gentleman's Diary, or Poetical and Mathematical Almanack for the Year 1861. London: Price one shilling and six pence.

This is the 158th annual number of this interesting and important publication. It is impossible to estimate the influence this unpretending serial has exerted upon the progress of the mathematics, and especially upon the development of mathematical talent in England. Besides 1988 original problems with their solutions, embracing all departments of the science, the mathematical papers thus far published would make several most valuable volumes. In our next we shall give the problems proposed for the next year, in hopes that many of our younger readers will feel disposed to send solutions of them "to the Editor of the Lady's and Gentleman's Diary, Stationer's Hall, London," post-paid, before May 1st, 1861.

Elements of the Differential and Integral Calculus. By WILLIAM SMYTH, A. M., Professor of Mathematics in Bowdoin College. Second Edition. Portland: Published by Sanborn and Carter. 1859.

The luminous statement, given in the Preface, of the spirit and plan of the work, at once arrests the attention; and an examination shows that the author is not only master of his subject, but clearly comprehends the manner of its presentation for the purpose of instruction. We have no hesitation in saying that this is one of the very best text-books upon the subject we have ever seen. The infinitesimal method of LEIBNITZ is adopted as its basis, which the author considers less logical than NEWTON'S Method of Limits or LAGRANGE'S Method of Derived Functions, but better adapted to the purposes of instruction, both on account of its greater simplicity, and the facility with which it is applied to the solution of the ordinary problems demanding the calculus, and especially for all purposes of investigation. Differentials are regarded simply as auxiliary quantities, and those who are familiar with the views of CARNOT (to whom our author acknowledges his indebtedness), as exhibited in his incomparable little work, entitled, *Reflexions sur la Métaphysique du Calcul Infinitésimal*, will at once understand the spirit of the work before us. The arrangement is excellent, as the following brief synopsis will show. I. First Principles. II. Derivation of the auxiliary quantities, or differentials. III. Application to problems in which the differentials can be elim-

inated by the ordinary processes of Algebra. IV. Integral Calculus, in which the auxiliaries, or differentials, are eliminated by reversing the process by which they were obtained. V. Application to problems of quadrature, cubature, rectification; surfaces of solids. VI. Successive differentials, or new auxiliaries. VII. Application of calculus to development of functions. VIII. Maxima and minima. IX. Theory of curves. X. Transcendental functions, their differentiation and integration. XI. Transcendental curves. XII. Processes of integration. XIII. Cycloid. XIV. Quadratures and cubatures continued. XV. Application to Mechanics. XVI. Equilibrium, Centre of Gravity, Hydrodynamics. XVII. Method of Variations. XVIII. Application of Calculus to Astronomy. XIX. Limits, Derived Functions. XX. Miscellaneous examples.

The clearness and simplicity with which the principles are developed, the interesting historical references, and the variety of the problems, all combine to make the work before us an excellent text-book, and as such we commend it to teachers and students, with the assurance that they will not be disappointed.

The methods of NEWTON and LAGRANGE are very properly given at the end of the work, because the student is then prepared to compare them with the method of LEIBNITZ without confusion.

Editorial Items.

WE are happy to lay before our readers the following report; and especially so, because all the Essays submitted are found of "sufficient merit to be worthy of publication." We shall print the Essays in the order named as rapidly as possible, and all in this volume of the Monthly.

REPORT OF THE COMMITTEE ON PRIZE ESSAYS.

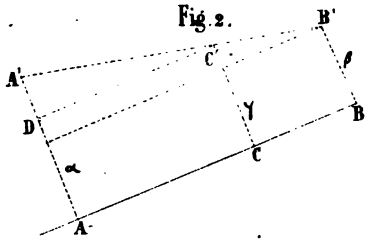
The Committee on Prize Essays regret that their Report has been so long delayed. The protracted absence of one of their number and the engagements of the rest prevented an earlier decision. They have received four Essays: On the Conformation of the Earth; On Central Forces; On Projections; and On Spherical Conics. All of these they consider of "sufficient merit to be worthy of publication;" although, as the productions of students, they are not to be too severely criticised. The Committee award the following prizes: To the first Essay, in the order named, forty dollars; to the second, thirty dollars; to the third, twenty dollars; and to the fourth, ten dollars.

WILLIAM FERREL,
J. B. HENCK,
CHAUNCEY WRIGHT.

BOOKS RECEIVED. — General Problems from the Orthographic Projections of Descriptive Geometry, &c., by PROF. S. EDWARD WARREN. (See Notice.) For sale by Brown and Taggard, Boston. — Annual Report of the Board of Regents of the Smithsonian Institution, showing the operations, expenditures, and condition of the Institution for the year 1859. — On the Equation of Differences for an Equation of any Order, and in particular for the Equations of the Orders Two, Three, Four, and Five, by ARTHUR CAYLEY, F. R. S.; also a paper on Recent Terminology in the Mathematics. — Progressive Intellectual, Practical, and Higher Arithmetics; New Elementary and University Algebras; New Plane and Solid Geometry, with Plane and Spherical Trigonometry; Surveying and Navigation, by H. N. ROBINSON, LL. D., published by Ivison, Phinney, & Co., 48 and 50 Walker Street, New York.

Fig. 1.

Fig. 2.



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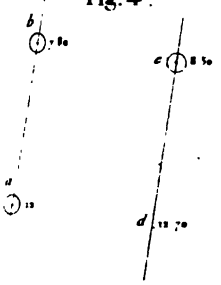

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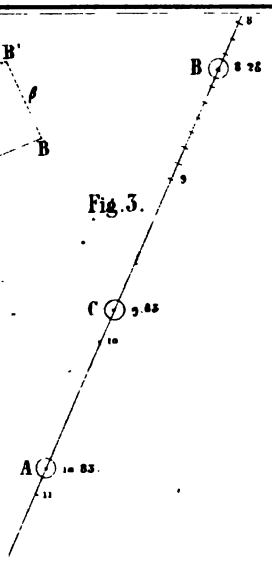

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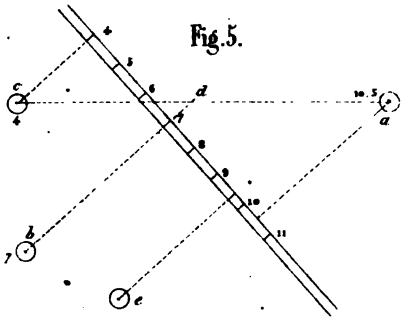

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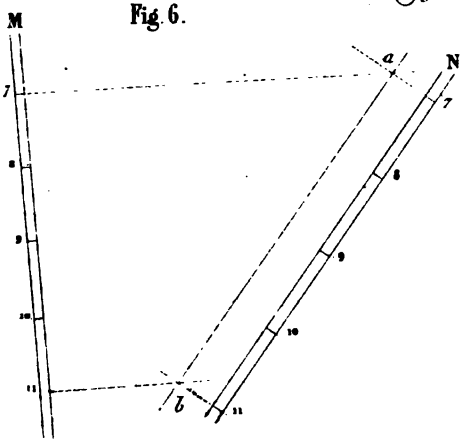

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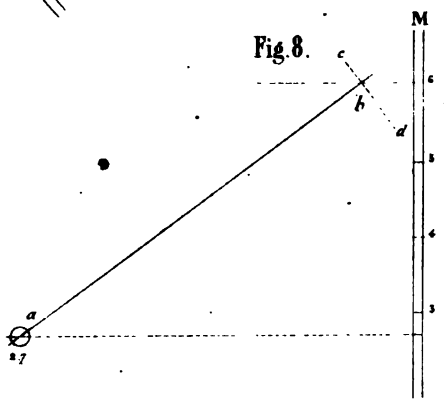

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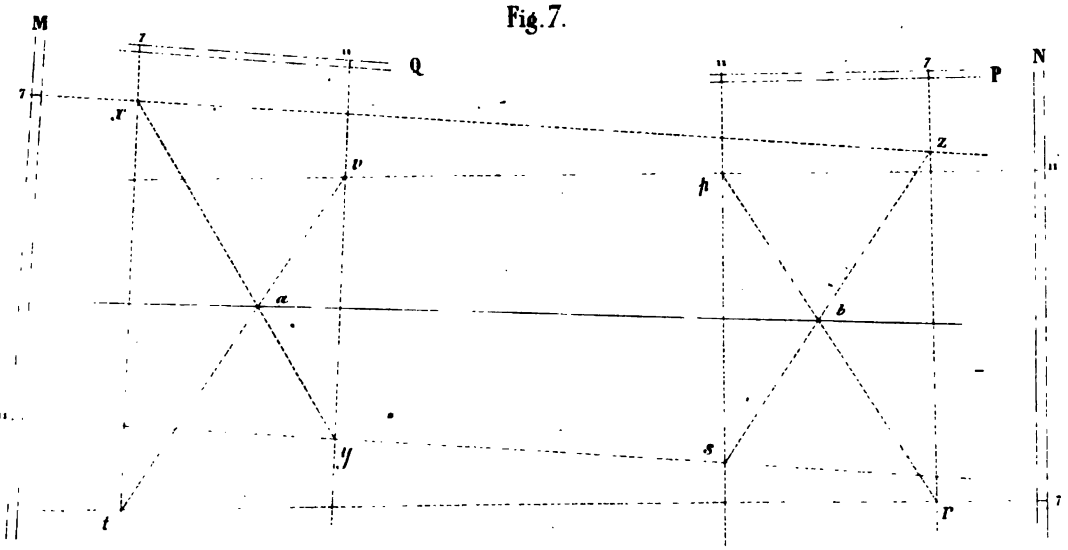

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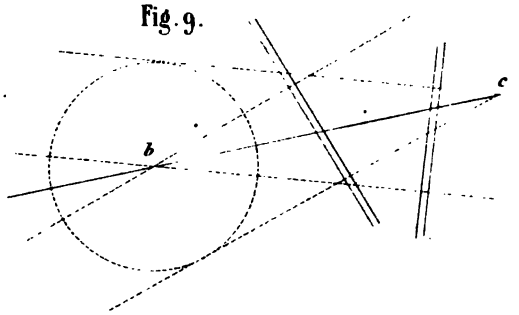
Fig. 9.



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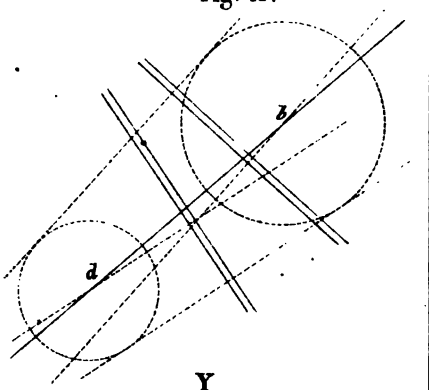

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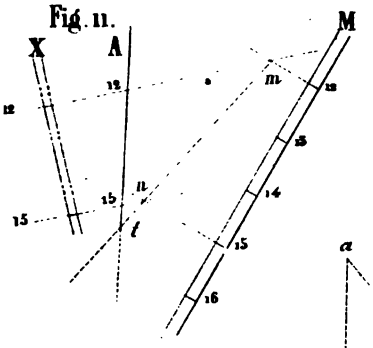

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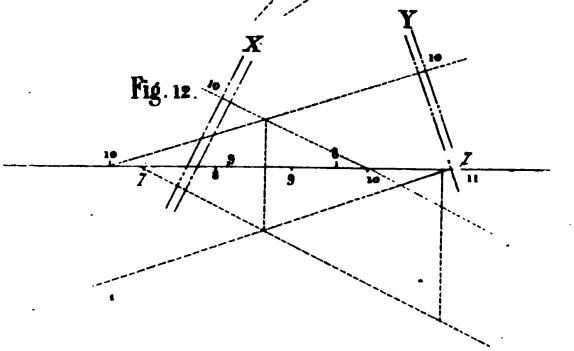

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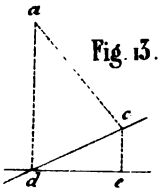

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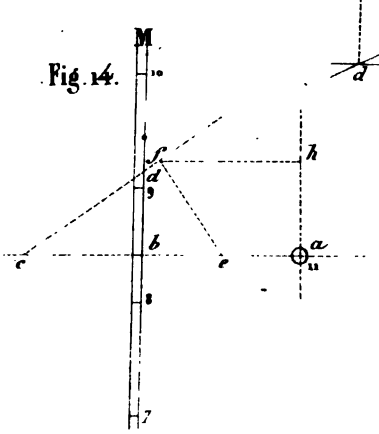

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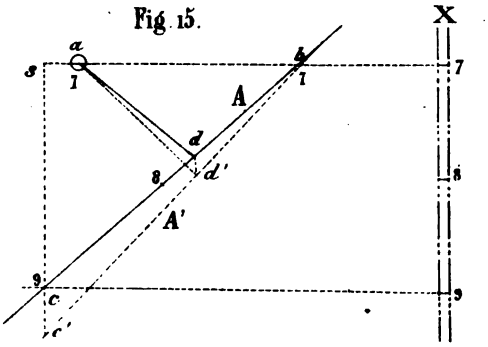


Fig. 16.

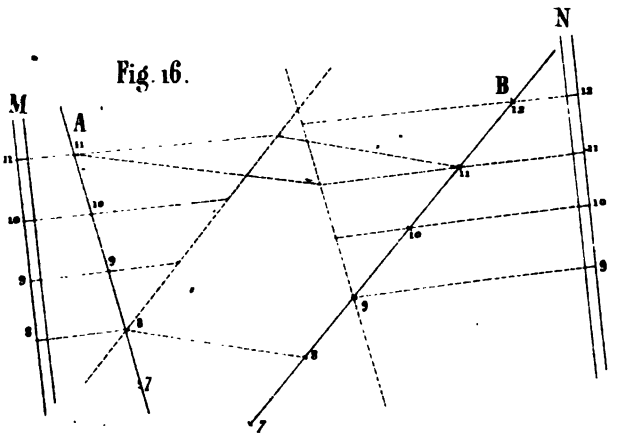


Fig. 17.

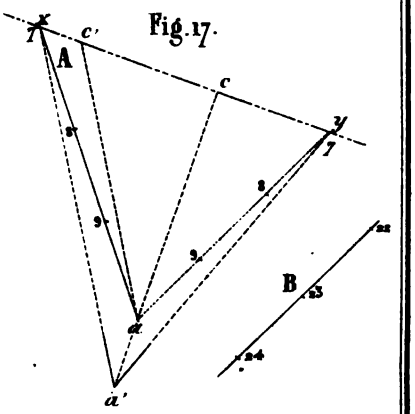


Fig.18.

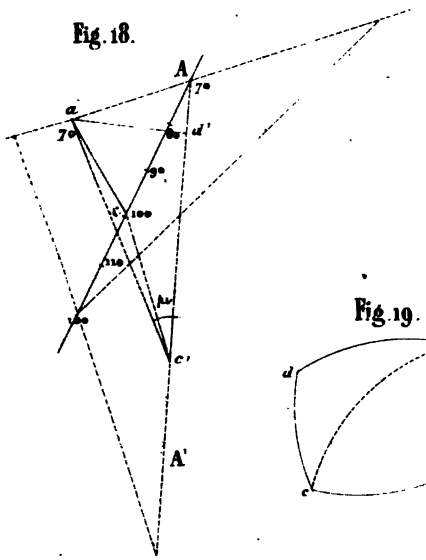


Fig.19.

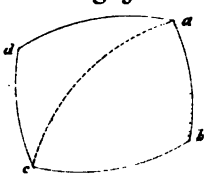


Fig.20.

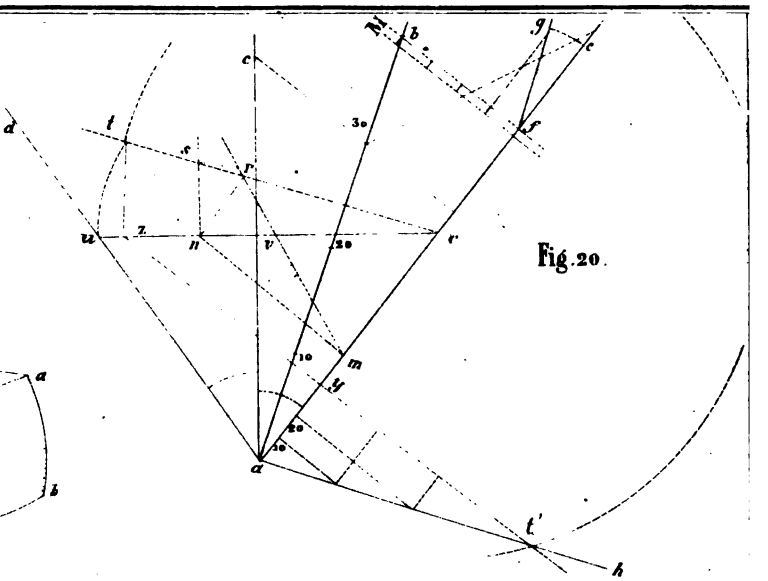


Fig.21.

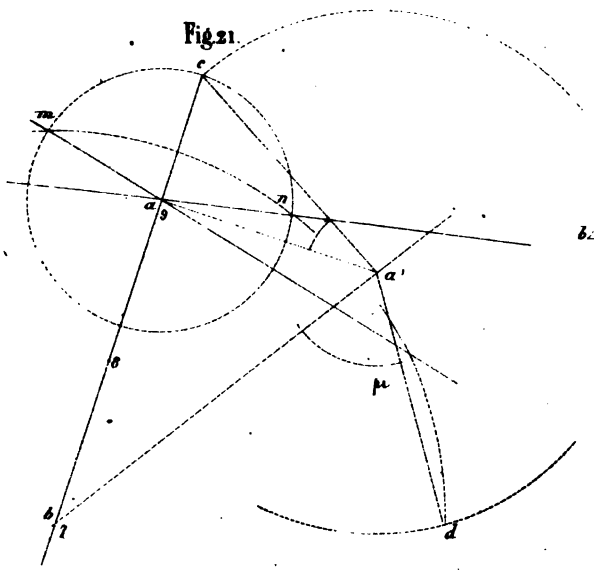


Fig.22.

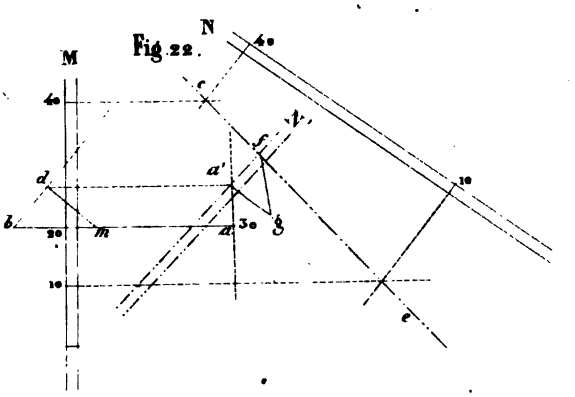


Fig.23.

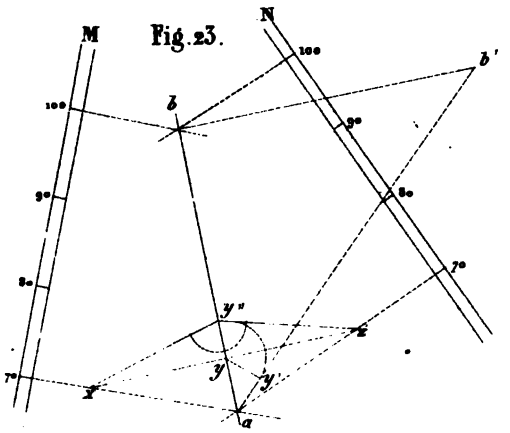
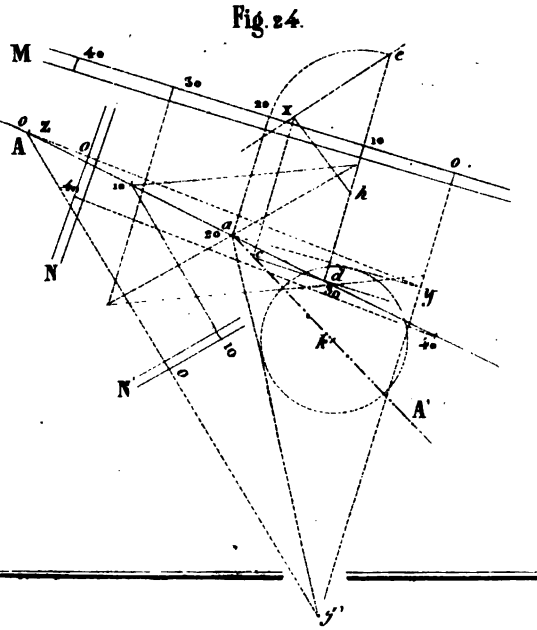


Fig.24.



Lith. Goyer 7 Pas Dauphine, Paris

Fig. 26.

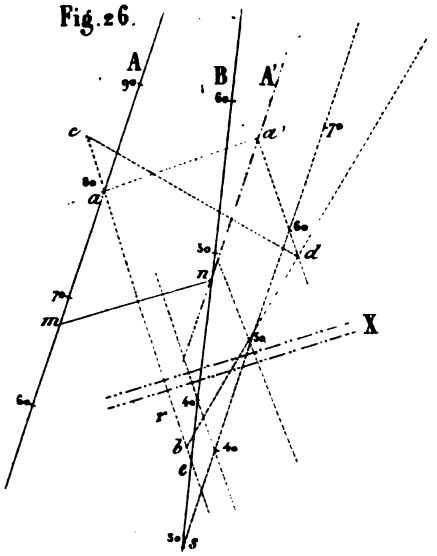


Fig. 25.

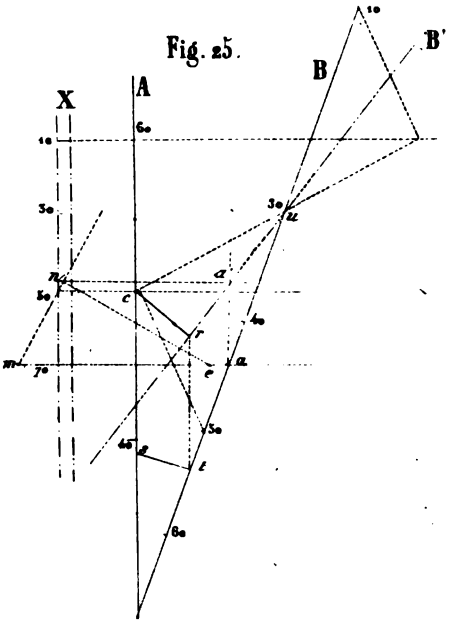


Fig. 27.

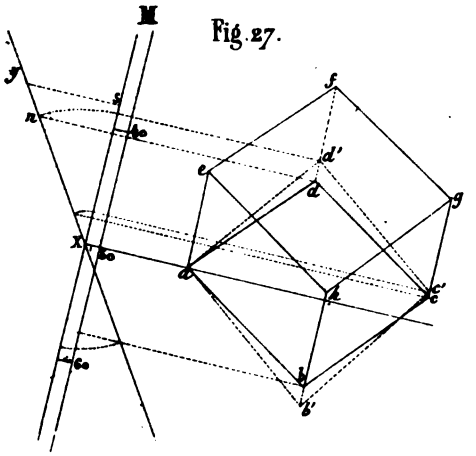


Fig. 28.

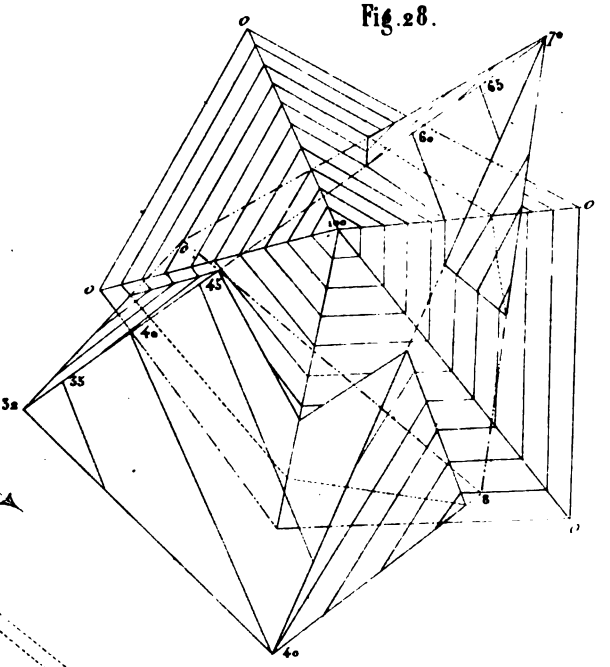
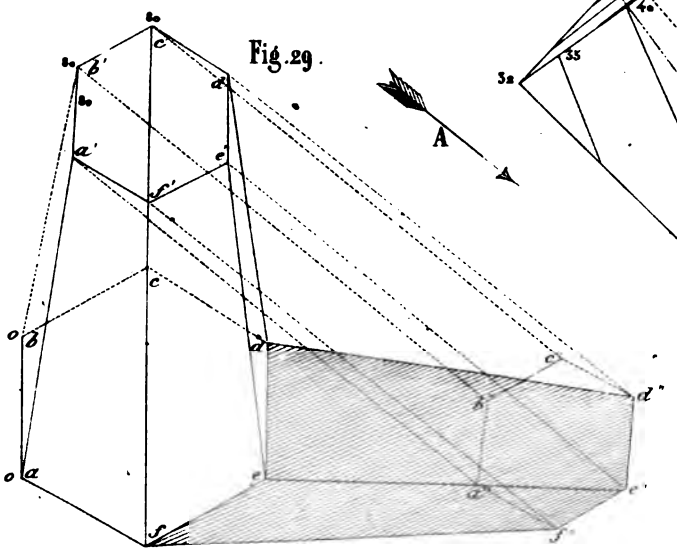


Fig. 29.



THE

MATHEMATICAL MONTHLY.

Vol. III. . . . MARCH, 1861. . . . No. VI.

PRIZE PROBLEMS FOR STUDENTS.

I. Given

$$(1) \quad a^2 - 3x^4 + (2+4y)x^3 + (4y+1-2a)x^2 - (2a+4ay)x = 4y^4,$$

$$(2) \quad x = b + y,$$

to find x and y by quadratics. — Communicated by Prof. E. W. Evans, Marietta College, Ohio.

II. Describe an isosceles triangle having each of the angles at the base double of the third angle.

III. Which is greater, $2\tan^{-1}(\sqrt{2}-1)$, or $3\tan^{-1}\frac{1}{4} + \tan^{-1}\frac{5}{99}$? — Communicated by Prof. Richard Cotter, Baltimore, Maryland.

IV. In any spherical triangle

$$\frac{\sin s \sin(s-a)\sin(s-b)\sin(s-c)}{\cos S \cos(S-A)\cos(S-B)\cos(S-C)} = \frac{\cos a \cos b \cos c - 1}{\cos A \cos B \cos C + 1},$$

in which $s = \frac{1}{2}(a+b+c)$, $S = \frac{1}{2}(A+B+C)$.*

V. If the earth be completely covered by a sea of small depth, prove that the depth in latitude l is very nearly equal to

$$H(1 - \varepsilon \sin^2 l),$$

where H is the depth at the equator, and ε the ellipticity of the earth.

Solutions of these problems must be received by May 1st, 1861.

* This problem was misprinted in No. XII., Vol. II.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS IN No. II., Vol. III.

THE first prize is awarded to GEORGE A. OSBORNE, Jr., Lawrence Scientific School, Cambridge, Mass.

The second prize is awarded to DAVID TROWBRIDGE, Perry City, N. Y.

The third prize is awarded to ASHER B. EVANS, Madison University, Hamilton, N. Y.

The fourth prize, to junior competitors, is not awarded.

PRIZE SOLUTION OF PROBLEM I.

Prove that the surface of a sphere can be completely covered with the surfaces either of 4, or of 8, or of 20 equilateral spherical triangles.

Suppose either a tetrahedron, an octahedron, or an icosahedron to be inscribed in a sphere. Then planes passing through the edges of the polyhedron and the centre of the sphere will divide its surface into either 4, 8, or 20 equilateral triangles. This solution is given by nearly all the competitors.

The solution of Problem II. by the usual formula is obvious.

PRIZE SOLUTION OF PROBLEM III.

By ASHER B. EVANS, Madison University, Hamilton, N. Y.

$$\frac{(x+a+b)^5+(x+c+d)^5}{(x+a+c)^5+(x+b+d)^5}=\frac{m}{n}.$$

Find x by quadratics.

Since the sum of the fifth powers of two numbers is divisible by the sum of the numbers, both numerator and denominator of the first member of the given equation is divisible by

$$2x+a+b+c+d.$$

Performing this division, and putting $P=2x+a+b+c+d$ in the result, we get

$$(1)\quad \frac{P^4-5P^2(x+a+b)(x+c+d)+5(x+a+b)^2(x+c+d)^2}{P^4-5P^2(x+a+c)(x+b+d)+5(x+a+c)^2(x+b+d)^2}=\frac{m}{n}.$$

But $x = \frac{P-a-b-c-d}{2}$, which, substituted in (1), gives, after reduction,

$$(2) \quad \frac{P^4+10\{(a+b)^2-(c+d)^2\}P^2+5\{(a+b)^2-(c+d)^2\}}{P^4+10\{(a+c)^2-(b+d)^2\}P^2+5\{(a+c)^2-(b+d)^2\}} = \frac{m}{n}.$$

By putting $p=(a+b)^2-(c+d)^2$, $q=(a+c)^2-(b+d)^2$, (2) becomes

$$(3) \quad P^4+\frac{10}{n-m}(np-mq)P^2=\frac{5}{n-m}(mq-np),$$

or

$$(4) \quad P^4+2AP^2=-A,$$

by putting, $\frac{5}{n-m}(np-mq)=A.$

From (4) we get the four roots,

$$P=\pm(-A\pm\sqrt{(A-\tfrac{1}{2})^2-\tfrac{1}{4}})^{\frac{1}{2}}.$$

These values of P, substituted in $x=\frac{P-a-b-c-d}{2}$, give four of the values of x, and the fifth is given by putting the divisor $2x+a+b+c+d=0$.

Therefore $$x=-\frac{a+b+c+d}{2}$$

is the fifth root.

W. D. MERSHON's solution is similar to the above.

PRIZE SOLUTION OF PROBLEM IV.

By DAVID TROWBRIDGE, Perry City, Schuyler County, N. Y.

A fraudulent tradesman contrives to employ his *false* balance both in buying and selling a certain article, thereby gaining at the rate of 11 per cent more on his outlay than he would gain if the balance were *true*. If however the scale-pans, in which the article is weighed when bought and sold respectively, were interchanged, he would neither gain nor lose by the article. Find the legitimate gain per cent on the article.

Let x denote the fraction of gain or loss in weight by using the false balance, and r the legitimate gain per cent. Every actual unit

of weight weighs $1+x$ when sold, and $1-x$ when bought; he sells each unit for $(1+r)(1+x)$, and buys it for $(1-x)$. His gain is therefore $(1+r)(1+x)-(1-x)$, and the gain per cent is

$$(1) \qquad \frac{(1+r)(1+x)-(1-x)}{1-x} = r + \tfrac{111}{100},$$

by using the false balance. By interchanging the scale-pans he would buy an actual unit of weight for $1+x$, and sell it for $(1+r)(1-x)$; therefore

$$(2) \qquad (1+r)(1-x)-(1+x) = 0,$$

since by the question he neither gains nor loses by the transaction.

Equations (1) and (2) give

$$(3) \qquad (1+r)\frac{(1+x)}{(1-x)} = 1+r+\tfrac{111}{100},$$

$$(4) \qquad \frac{1+x}{1-x} = 1+r.$$

Combining (3) and (4) we get

$$(1+r)^2-(1+r) = \tfrac{111}{100}; \qquad \therefore r = \tfrac{1}{10},$$

or 10 per cent is the legitimate gain.

Prize Solution of Problem V.

By George A. Osborne, Jr., Lawrence Scientific School.

Show that in the parabola the half sum of the radii vectores drawn to the extremities of any arc whatever, is equal to the radius vector drawn to the summit of the diameter passing through the centre of the chord and parallel to the axis, plus the part of this diameter intercepted between the arc and the chord.

Let CD be any arc of the parabola, and F the focus. From C and D, the extremities of the chord CD, and also from its middle point, N, drop perpendiculars, DO, NP, and CQ, upon the directrix AB; and let M be the point in which NP meets the curve. Draw the radii FC, FM, and FD; then by the known property, that any

point in the curve is equally distant from the focus and directrix, we have

$$PN = \tfrac{1}{2}(OD + CQ) = \tfrac{1}{2}(DF + CF)$$
$$= PM + MN = MF + MN;$$
$$\therefore MF + MN = \tfrac{1}{2}(DF + CF).$$

Several of the competitors gave substantially this solution.

Mr. G. B. Hicks gave the following solution: —

Let the abscissas of the extremities of any arc be designated by x_1 and x_2, and the radii vectores drawn to these extremities by r_1 and r_2.

Then calling the equation to the curve $y^2 = 4mx$, we have, from the fundamental property of the parabola,

$$r_1 = x_1 + m, \qquad r_2 = x_2 + m,$$
$$\therefore \tfrac{1}{2}(r_1 + r_2) = \tfrac{1}{2}(x_1 + x_2) + m.$$

Again, let the radius vector drawn to the summit of the diameter passing through the centre of the chord and parallel to the axis be denoted by r_3, and the abscissa of the summit by x_3.

And let h equal the part of the diameter intercepted between the arc and the chord. We have

$$r_3 = x_3 + m.$$

And since the abscissa of the centre of the chord is $\frac{1}{2}(x_1 + x_2)$,

$$h = \tfrac{1}{2}(x_1 + x_2) - x_3;$$
$$\therefore r_3 + h = \tfrac{1}{2}(x_1 + x_2) + m = \tfrac{1}{2}(r_1 + r_2).*$$

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* One member of the Committee for awarding these Prizes has resigned. We shall hereafter attend to the duty, as it is often inconvenient for the Committee to report in time for the press. Besides, we have no right to demand the continuance of a serious labor so long and so ably performed. — Ed.

FIRST PRIZE ESSAY.—ON THE CONFORMATION OF THE EARTH.

By G. W. Hill, Nyack, N. Y.

1. All the particles which compose the mass of the earth are animated by the attraction of gravitation. The law of this force is, that the attraction of any atom for a spherical surface of material points, described about it as a centre, is constant. Hence, if the attraction of an atom for a material point be represented by A, and r be the radius of the spherical surface, and N the number of material points in a unit of surface, the attraction of the central atom for the spherical surface is $4\pi N r^2 A =$ a constant $= -4\pi N M$. Whence $A = -\frac{M}{r^2}$; that is, the attraction varies inversely as the distance squared. The constant M is called the mass of the attracting atom. We have given A a negative sign because it represents a force tending to decrease the line r.

2. Making $A = \frac{dV}{dr}$, then $V = \frac{M}{r}$. V is called the *potential function*, and has this property: that if the partial differential of it be taken with respect to any of the three spaceal co-ordinates of which it is necessarily a function, the result will be the partial force in the direction of that co-ordinal axis.

3. If the attraction of a single atom give $V = \frac{M}{D}$, denoting the distance by D, the attraction of an indefinite number or assemblage of atoms will give $V = S \cdot \frac{M}{D}$. If ξ, ψ, φ represent any three lines at right angles with each other, then $\frac{dV}{d\xi}, \frac{dV}{d\psi}, \frac{dV}{d\varphi}$ are the forces acting in each of these three directions respectively. In a rectangular system $D = \{(x-x')^2 + (y-y')^2 + (z-z')^2\}^{\frac{1}{2}}$, the accented letters pertain to the attracting atom, the unaccented to the attracted. Consequently,

$$(1)\ \frac{dV}{dx} = S.-\frac{M}{D^3}(x-x'),\ \frac{dV}{dy} = S.-\frac{M}{D^3}(y-y'),\ \frac{dV}{dz} = S.\frac{M}{D^3}(z-z').$$

4. Differentiating again,

$$(2)\qquad \begin{cases} \frac{d^2V}{dx^2} = S \cdot \frac{M}{D^3}\left\{\frac{3}{D^2}(x-x')^2 - 1\right\}, \\ \frac{d^2V}{dy^2} = S \cdot \frac{M}{D^3}\left\{\frac{3}{D^2}(y-y')^2 - 1\right\}, \\ \frac{d^2V}{dz^2} = S \cdot \frac{M}{D^3}\left\{\frac{3}{D^2}(z-z')^2 - 1\right\}. \end{cases}$$

In this differentiation M has been regarded as independent of x, y, z; but in order to render equations (2) altogether general, the attracting mass must be considered as extending into the point x, y, z. Let ϱ be the density of the atom occupying this point, which becomes zero when the attracting mass does not reach the point. This atom may be regarded as spherical, then for it $M = \frac{4\pi\varrho D^3}{3}$; substituting this value in equations (1), the results are

$$\frac{dV}{dx} = \frac{4\pi\varrho}{3}(x'-x), \quad \frac{dV}{dy} = \frac{4\pi\varrho}{3}(y'-y), \quad \frac{dV}{dz} = \frac{4\pi\varrho}{3}(z'-z).$$

Hence we must add the term $-\frac{4\pi\varrho}{3}$ to the right members of equations (2), and then we can regard D as having always a finite value. By adding these equations, there results

$$(3)\qquad \frac{d^2V}{dx^2} + \frac{d^2V}{dy^2} + \frac{d^2V}{dz^2} + 4\pi\varrho = 0.$$

The integration of this gives V; ϱ is a function of x, y, z; in the case of solid bodies, as the earth, a limited function.

5. Transform (3) to terms of polar co-ordinates; put

$$x = r\sqrt{(1-\mu^2)}\cos\omega, \qquad r = \sqrt{(x^2+y^2+z^2)},$$
$$y = r\sqrt{(1-\mu^2)}\sin\omega, \qquad \mu = \frac{z}{\sqrt{(x^2+y^2+z^2)}},$$
$$z = r\mu, \qquad \omega = \tan^{-1}\frac{y}{x}.$$

Then

$$\frac{d^2V}{dx^2}+\frac{d^2V}{dy^2}+\frac{d^2V}{dz^2}=\left\{\frac{dr^2}{dx^2}+\frac{dr^2}{dy^2}+\frac{dr^2}{dz^2}\right\}\frac{d^2V}{dr^2}+\left\{\frac{d^2r}{dx^2}+\frac{d^2r}{dy^2}+\frac{d^2r}{dz^2}\right\}\frac{dV}{dr}$$
$$+\left\{\frac{d\mu^2}{dx^2}+\frac{d\mu^2}{dy^2}+\frac{d\mu^2}{dz^2}\right\}\frac{d^2V}{d\mu^2}+\left\{\frac{d^2\mu}{dx^2}+\frac{d^2\mu}{dy^2}+\frac{d^2\mu}{dz^2}\right\}\frac{dV}{d\mu}$$
$$+\left\{\frac{d\omega^2}{dx^2}+\frac{d\omega^2}{dy^2}+\frac{d\omega^2}{dz^2}\right\}\frac{d^2V}{d\omega^2}+\left\{\frac{d^2\omega}{dx^2}+\frac{d^2\omega}{dy^2}+\frac{d^2\omega}{dz^2}\right\}\frac{dV}{d\omega},$$
$$=r^{-2}\left\{\frac{d.r^2\frac{dV}{dr}}{dr}+\frac{d.(1-\mu^2)\frac{dV}{d\mu}}{d\mu}+\frac{\frac{d^2V}{d\omega^2}}{1-\mu^2}\right\}.$$

Hence (3) becomes

$$(4)\qquad \frac{d.r^2\frac{dV}{dr}}{dr}+\frac{d.(1-\mu^2)\frac{dV}{d\mu}}{d\mu}+\frac{\frac{d^2V}{d\omega^2}}{1-\mu^2}+4\pi\varrho r^2=0.$$

6. To show the application of (4), take the simple case when the surfaces of equal density are concentrically spherical. Placing the origin of co-ordinates at the common centre, ϱ becomes a function of r alone, either continuous or discontinuous, as the case demands; and evidently $\frac{dV}{d\mu}=0$, $\frac{dV}{d\omega}=0$; therefore (4) becomes

$$\frac{d.r^2\frac{dV}{dr}}{dr}+4\pi\varrho r^2=0.$$

By integration, $\frac{dV}{dr}=(C-4\pi\int\varrho r^2\,dr)r^{-2}$.

$4\pi\int\varrho r^2\,dr$, between the limits 0 and r, is equal to the mass contained within the sphere whose radius is r; denoting this by M, it is clear that $C=0$, since the expression must agree with that for the attraction of a single atom. Thus $\frac{dV}{dr}=-\frac{M}{r^2}$. Or the principle may be stated: "The force acting on any point, wherever situated, equals the mass of all the particles nearer the nucleal centre than the point attracted, divided by the square of the point's distance from that centre, with the negative sign."

7. The earth revolves uniformly about a constant axis; hence, to remove our problem from dynamics to statics, it is necessary to in-

troduce the force of pressure called the centrifugal force. Making the co-ordinal axis of z coincide with the earth's axis, and T denoting the period of its revolution, the potential of the centrifugal force is

$$V = \frac{2\pi^2}{T^2}(x^2+y^2) = \frac{2\pi^2}{T^2} r^2(1-\mu^2).$$

Since this force animates every particle, include its potential in V, and make V the potential of both gravitating and centrifugal force. It then becomes necessary to add to (4) the term $-\frac{8\pi^2}{T^2} r^2$,

$$(5) \quad \frac{d \cdot r^2 \frac{dV}{dr}}{dr} + \frac{d \cdot (1-\mu^2) \frac{dV}{d\mu}}{d\mu} + \frac{\frac{d^2V}{d\omega^2}}{1-\mu^2} + 4\pi\left(\rho - \frac{2\pi}{T^2}\right) r^2 = 0.$$

8. To apply this equation to the solution of the earth's conformation, we must combine it with some condition of equilibrium. From the manner in which the atmosphere and ocean cover the earth, we may conjecture it was once fluid, and in solidifying preserved the form it had taken by the laws of hydrostatics. In passing from the solid earth to the ocean, and from the ocean to the atmosphere, there occur two faults in the continuity of the earth's density; hence ρ is strictly represented by a discontinuous function. But as the mass of the ocean and atmosphere is about $\frac{1}{4000}$ of the whole, its influence may be neglected, and ρ supposed continuous from centre to surface.

9. If p is the pressure, then $dp = \rho\, dV$, and $V + B = 0$, is the equation to surfaces of level, B having a different value for every surface. Let ρ be a function of p, and thus of V. In (5) make

$$4\pi\left(\rho - \frac{2\pi}{T^2}\right) = f(V).$$

10. The centrifugal force being small compared with gravity, may be regarded as a perturbing force. Supposing at first this force is zero, the particles would arrange themselves symmetrically about a centre,

since there is no reason why they should accumulate in one place rather than in another. Take the origin of co-ordinates at this centre, then $\frac{dV}{d\mu} = 0$, $\frac{dV}{d\omega} = 0$. Thus (5) becomes

$$\frac{d.r^2 \frac{dV}{dr}}{dr} + r^2 f(V) = 0.$$

Consequently V is a function of r alone, and the general equation to surfaces of level, $V + B = 0$, when solved gives $r =$ a constant; these surfaces are, then, concentrically spherical.

11. The term $\frac{2\pi^2}{T^2} r^2 (1 - \mu^2)$, which the centrifugal force introduces into V, and which causes a departure from the spherical form, does not contain ω, and so whatever derangement it may produce, cannot introduce ω into V; that is, the earth is a solid of revolution. Consequently (5) becomes

$$(6) \qquad \frac{d.r^2 \frac{dV}{dr}}{dr} + \frac{d.(1-\mu^2) \frac{dV}{d\mu}}{d\mu} + r^2 f(V) = 0.$$

12. From the form of this same term it may be concluded that

$$(7) \qquad V = Y_0 + Y_1 \mu^2 + Y_2 \mu^4 + \&c. \ldots = \Sigma . Y_i \mu^{2i},$$

where Y_i is a function of r alone, and a quantity of the order of the ith power of the centrifugal force. Substitute this expression of V in (6) and put the coefficient of μ^{2i} resulting equal to zero, and let the coefficient of μ^{2i} in $r^2 f(V)$ be U_i, then

$$(8) \quad \frac{d.r^2 \frac{dY_i}{dr}}{dr} + (2i+1)(2i+2) Y_{i+1} - 2i(2i+1) Y_i + U_i = 0.$$

This equation has the inconvenience of introducing Y_{i+1}; let us therefore assume more generally $V = \Sigma . V_i M_i$, V_i a function of r, and M_i of μ, V_i being of the same order as Y_i. Substituting in (6),

$$\Sigma . \frac{d.r^2 \frac{dV_i}{dr}}{dr} M_i + \Sigma . V_i \frac{d.(1-\mu^2) \frac{dM_i}{d\mu}}{d\mu} + r^2 f(\Sigma . V_i M_i) = 0,$$

which may be written

$$(9) \qquad \Sigma . \frac{d . r^2 \frac{dV_i}{dr}}{dr} M_i + \Sigma . V_i \frac{d . (1-\mu^2) \frac{dM_i}{d\mu}}{d\mu} + \Sigma . T_i M_i = 0.$$

In order that the left-hand member may be arranged in a series of the same form as $\Sigma . V_i M_i$, we must have

$$\frac{d . (1-\mu^2) \frac{dM_i}{d\mu}}{d\mu} = n M_i,$$

in which n is independent of μ. We may determine n from the consideration that V_i being of the same order as Y_i, M_i cannot contain any higher power of μ than μ^{2i}. Making $M_i = \Sigma . k_s \mu^{2s}$, this relation results: $k_{s+1} = \frac{n + 2s(2s+1)}{(2s+1)(2s+2)} k_s$. To make this series end at k_i, n must equal $-2i(2i+1)$; and $k_{s+1} = -\frac{(2i-2s)(2i+2s+1)}{(2s+1)(2s+2)} k_i$; hence, putting $k_0 = 1$, which is allowable,

$$(10) \quad M_i = 1 - \frac{2i(2i+1)}{1.2} \mu^2 + \frac{(2i-2)2i(2i+1)(2i+3)}{1.2.3.4} \mu^4 \ldots \pm \frac{2 \ldots 2i(2i+1) \ldots (4i+1)}{1.2.3 \ldots 2i} \mu^{2i}.$$

For V_i we have from (9), by rejecting the sign Σ and dividing by M_i, this equation: —

$$(11) \qquad \frac{d . r^2 \frac{dV_i}{dr}}{dr} - 2i(2i+1) V_i + T_i = 0.$$

From the expression (10) we easily deduce

$$(12) \quad Y_i = \pm \frac{(2i+1)(2i+3) \ldots (4i-1)}{1.3 \ldots (2i-1)} \left\{ V_i + \frac{(i+1)(4i+1)}{1.(2i+1)} V_{i+1} + \frac{(i+1)(i+2)(4i+1)(4i+3)}{1.2.(2i+1)(2i+3)} V_{i+2} + \&c. \ldots \right\}.$$

The reversion of which is

$$(13) \quad V_i = \pm \frac{1.3 \ldots (2i-1)}{(2i+1)(2i+3) \ldots (4i-1)} \left\{ Y_i + \frac{(i+1)(2i+1)}{1.(4i+3)} Y_{i+1} + \frac{(i+1)(i+2)(2i+1)(2i+3)}{1.2.(4i+3)(4i+5)} Y_{i+2} + \&c. \ldots \right\}.$$

The upper sign is to be used when i is even, the lower when odd. From (10) we obtain

$$M_1^2 = \tfrac{1}{5} - \tfrac{4}{7} M_1 + \tfrac{3}{3}\tfrac{7}{5} M_2. \tag{14}$$

From this and the equation $r^2 f(\Sigma . V_i M_i) = \Sigma . T_i M_i$, pursuing the approximation to quantities of the second order, we get these expressions for T_i,

$$\begin{cases} r^{-2} T_0 = f(V_0) + \tfrac{2}{5} V_1^2 f''(V_0) + \text{\&c.} \ldots \\ r^{-2} T_1 = V_1 f'(V_0) - \tfrac{3}{7} V_1^2 f''(V_0) + \text{\&c.} \ldots \\ r^{-2} T_2 = V_2 f'(V_0) + \tfrac{2}{7}\tfrac{7}{0} V_1^2 f''(V_0) + \text{\&c.} \ldots \end{cases} \tag{15}$$

If quantities of the second order are neglected, the two differential equations to be integrated are

$$\begin{cases} \frac{d . r^2 \frac{d V_0}{dr}}{dr} + r^2 f(V_0) = 0, \\ \frac{d . r^2 \frac{d V_1}{dr}}{dr} - 6 V_1 + r^2 V_1 f'(V_0) = 0. \end{cases} \tag{16}$$

To pursue the analysis farther would require a knowledge of the form of $f(V)$.

13. However, when the point r, μ, ω is without the surface of the earth, (11) can be integrated. Supposing the point not to partake of the motion of revolution, the centrifugal force must be neglected, and since $\rho = 0$, generally $T_i = 0$; consequently (11) becomes

$$\frac{d . r^2 \frac{d \breve{V}_i}{dr}}{dr} - 2i(2i+1) \breve{V}_i = 0. \tag{17}$$

The integral of this is $\breve{V}_i = a_i r^{2i} + b_i r^{-2i+1}$, in which $a_i = 0$; otherwise the earth's attraction would be infinitely great at an infinite distance. From the equation $V = \Sigma . V_i M_i$, by giving i successively the values 0, 1, 2, &c., we have, then,

$$\breve{V} = b_0 r^{-1} + b_1 r^{-3}(1 - 3\mu^2) + b_2 r^{-5}(1 - 10\mu^2 + \tfrac{3}{3}\tfrac{5}{5}\mu^4) + \text{\&c.} \ldots \tag{18}$$

We have written V in this case $\check{V}$. If r, μ, ω is situated in the atmosphere or at the surface, and revolves with the earth, we have only to add to $\check{V}$ the potential of the centrifugal force $\frac{2\pi^2}{T^2} r^2 (1-\mu^2)$, and may then call it $\bar{V}$; and for points within write $\hat{V}$.

14. The integration of (11) introduces an indefinite number of constants, of which the superfluous are to be eliminated in the usual way of treating partial differential equations, viz.: by equations of limitation; which in the present question are the equations having places at the limit of the attracting mass. At this limit $\frac{dV}{dr}$, $\frac{dV}{d\mu}$, $\frac{dV}{d\omega}$, derived from the integration of (5), must have the same value, whether ρ in that equation equals zero or its value at the surface. Consequently we have

$$(19) \qquad \frac{d.(\hat{V}-\bar{V})}{dr} = 0, \qquad \frac{d.(\hat{V}-\bar{V})}{d\mu} = 0, \qquad \frac{d.(\hat{V}-\bar{V})}{d\omega} = 0.$$

Hydrostatics furnishes two other equations,

$$(20) \qquad \hat{V} + B = 0, \qquad \bar{V} + B' = 0.$$

These five are all equations to the same surface, of which it must be noticed that they are not all independent, but any one is a consequence of the other four. They will give all the conditions needed to eliminate the superfluous arbitrary constants. The last of (19) is, since V does not involve ω, an identical equation; select then the first of (19) and the two of (20) for the purpose of elimination. It is evident from $V = \Sigma . V_i M_i$, that each one of these equations to the surface can be transformed to this shape: $r = \Sigma . c_i M_i$, c_i a constant and of the ith order with respect to the centrifugal force. By substituting this value of r in equations (19) and (20), and equating the coefficients of M_i generally to zero, and rejecting the two involving B and B', we have, pursuing the approximation to the ith order, $3i + 1$ equations. The constants they involve are the con-

stants resulting from the integration of (11), $2i+2$ in number, and the constants c_i, $i+1$ in number, in all $3i+3$. Thus two constants are left indeterminate, which is as it should be, since the units of length and density are arbitrary. We may assume, then, $c_0 = 1$; making this substitution in $\bar{V} + B = 0$, using accents to denote differentiation and putting r afterwards equal to unity, and taking account of quantities of the second order, we have

$$(21) \qquad \begin{cases} V_0' c_1 + V_1 - \frac{2}{7} c_1 \{ V_0'' c_1 + 2 V_1' \} = 0. \\ V_0' c_2 + V_2 + \frac{2}{70} c_1 \{ V_0'' c_1 + 2 V_1' \} = 0. \end{cases}$$

Treating $\bar{V} + B' = 0$ in the same way, two similar equations result, which can be evidently derived from (21) by making

$$V_0 = b_0 r^{-1} + \frac{4\pi^2}{3T^2} r^2, \quad V_1 = b_1 r^{-3} + \frac{2\pi^2}{3T^2} r^2, \quad \text{and} \quad V_2 = b_2 r^{-5},$$

$$(22) \qquad \begin{cases} \left\{ -b_0 + \frac{8\pi^2}{3T^2} \right\} c_1 + b_1 + \frac{2\pi^2}{3T^2} - \frac{2}{7} c_1 \left\{ 2 b_0 c_1 - 6 b_1 + \frac{8\pi^2}{3T^2} \right\} = 0, \\ -b_0 c_2 + b_2 + \frac{2}{70} c_1 \left\{ 2 b_0 c_1 - 6 b_1 + \frac{8\pi^2}{3T^2} \right\} = 0. \end{cases}$$

From the equation $\frac{d.(\bar{V} - V)}{dr} = 0$, also we have

$$(23) \qquad \begin{cases} V_0' + \frac{2}{5} c_1 \{ V_0''' c_1 + 2 V_1'' \} = -b_0 + \frac{8\pi^2}{3T^2} + \frac{2}{5} c_1 \left\{ -6 b_0 c_1 + 24 b_1 + \frac{8\pi^2}{3T^2} \right\}, \\ V_0'' c_1 + V_1' - \frac{2}{7} c_1 \{ V_0''' c_1 + 2 V_1'' \} = \left(2 b_0 + \frac{8\pi^2}{3T^2} \right) c_1 - 3 b_1 + \frac{4\pi^2}{3T^2} \\ \qquad\qquad - \frac{2}{7} c_1 \left\{ -6 b_0 c_1 + 24 b_1 + \frac{8\pi^2}{3T^2} \right\}, \\ V_0'' c_2 + V_2' + \frac{2}{70} c_1 \{ V_0''' c_1 + 2 V_1'' \} = 2 b_0 c_2 - 5 b_2 \\ \qquad\qquad + \frac{2}{70} c_1 \left\{ -6 b_0 c_1 + 24 b_1 + \frac{8\pi^2}{3T^2} \right\}. \end{cases}$$

Equations (22) and (23) can be reduced to the following simpler forms: —

$$(24)\quad \begin{cases} b_0 = -\{1 + \frac{2}{3}q + \frac{1}{5}q^2 - \frac{4}{3}q\,c_1 + \frac{38}{5}c_1^2\}\ \{V_0' + \frac{2}{5}c_1\,[V_0'''\,c_1 + 2\,V_1'']\}, \\ b_1 = \left\{(1 - \frac{4}{21}q)\,c_1 - \frac{2}{7}c_1^2 - \frac{q}{6}\left(1 - \frac{q}{2}\right)\right. b_0, \\ b_2 = \{c_2 - \frac{9}{14}q\,c_1 + \frac{54}{35}c_1^2\}\,b_0, \end{cases}$$

$$(25)\quad \begin{cases} V_0''\,c_1 + V_1' - \frac{2}{7}c_1\{V_0'''\,c_1 + 2\,V_1''\} - \{(1 - \frac{32}{21}q)\,c_1 + \frac{12}{7}c_1^2 - \frac{5}{6}q\left((1 + \frac{q}{6})\right\}V_0' = 0 \\ V_0''\,c_2 + V_2' + \frac{27}{70}c_1\{V_0'''\,c_1 + 2\,V_1''\} - \{3\,c_2 - \frac{27}{14}q\,c_1 + \frac{27}{35}c_1^2\}\,V_0' = 0, \end{cases}$$

if we make $\frac{4\,n^2}{T^2} = q\,\{b_0(1 - 3\,c_1) + 3\,b_1\}$.

15. In this article we shall neglect quantities of the order of q^2. Let $\frac{w}{3}$ denote the quantity obtained by dividing the second member of the second equation of (23) by the second member of the first equation, then

$$w = 3\,.\,\frac{-3\,b_1 + \frac{4\,\pi^2}{3\,T^2} + 2\,b_0\,c_1}{b_0} = 6\,c_1 + q - 9\,\frac{b_1}{b_0};$$

from the second equation of (24), $3\,c_1 = \frac{3\,b_1}{b_0} + \frac{q}{2} = e$, the approximate value of the earth's compression, as is clear from the equation $r = 1 + c_1(1 - 3\,\mu^2) + $ &c.... Hence by addition

$$(26)\qquad e + w = 6\,c_1 + \tfrac{3}{2}\,q - 6\,\frac{b_1}{b_0} = \tfrac{5}{2}\,q.$$

The relation enunciated by this equation is known as CLAIRAUT'S Theorem.

16. The equation of the earth's surface is

$$\begin{aligned} r &= 1 + c_1(1 - 3\,\mu^2) + c_2(1 - 10\,\mu^2 + \tfrac{35}{3}\,\mu^4) + \text{&c.}... \\ &= (1 + c_1 + c_2)\{1 - [3\,c_1(1 - c_1) + 10\,c_2]\,\mu^2 + \tfrac{35}{3}\,c_2\,\mu^4 + \text{&c.}...\} \\ (27)\qquad &= a\{1 + e_1\sin^2\theta + e_2\sin^4\theta + \text{&c.}...\}; \end{aligned}$$

in which θ is the geocentric latitude. If e represent the compression, or part of its own length by which the equatorial radius exceeds the polar,

$$(28)\qquad e = -e_1 - e_2 = 3\,c_1(1 - c_1) - \tfrac{5}{3}\,c_2.$$

17. Denoting the normal or astronomical latitude by θ', we have

$$(29)\quad \theta' = \theta - \tan^{-1}\frac{dr}{r\,d\theta} = \theta - \left\{e_1 + e_2 - \frac{e_1^2}{2}\right\}\sin 2\theta - \frac{2e_2 - e_1^2}{4}\sin 4\theta,$$

the reversion of which is

$$(30)\qquad \theta = \theta' + \left\{e_1 + e_2 - \frac{e_1^2}{2}\right\}\sin 2\theta' + \frac{2e_2 + 3e_1^2}{4}\sin 4\theta'.$$

18. A line geodetically measured on the earth's surface is clearly the shortest possible; hence, if ds is the element of the curve, by the principles of the Calculus of Variations, $d.\int ds + \lambda\,\delta V = 0$, λ being the indeterminate multiplier of δV. Also,

$$ds^2 = dr^2 + \frac{r^2\,d\mu^2}{1-\mu^2} + r^2(1-\mu^2)\,d\omega^2.$$

The coefficients of δr, $\delta\mu$, $\delta\omega$ each equal zero; retaining only that of $\delta\omega$, as sufficient for our purpose, we have

$$d.\left\{r^2(1-\mu^2)\frac{d\omega}{ds}\right\} = 0, \quad \text{or} \quad r^2(1-\mu^2)\frac{d\omega}{ds} = ah.$$

That is, the sine of the angle made by this curve with the meridians varies inversely as the distance from the earth's axis. Hence

$$\frac{d\omega}{d\mu} = \frac{ah\sqrt{1+\left(\frac{dr}{r\,d\mu}\right)^2(1-\mu^2)}}{(1-\mu^2)\sqrt{r^2(1-\mu^2)-a^2h^2}} \quad \text{and} \quad \frac{ds}{d\mu} = \frac{r^2\sqrt{1+\left(\frac{dr}{r\,d\mu}\right)^2(1-\mu^2)}}{\sqrt{r^2(1-\mu^2)-a^2h^2}}.$$

Taking account only of the first order with respect to q,

$$d\omega = \frac{h\,d\mu}{(1-\mu^2)\sqrt{(1-h^2-\mu^2)}} - \frac{h\,e_1\,\mu^2\,d\mu}{(1-h^2-\mu^2)^{\frac{3}{2}}}.$$

$$(31)\quad \omega = C - \sin^{-1}\frac{1-\frac{1+h^2}{1-h^2}\mu^2}{1-\mu^2} + h\,e_1\left\{\sin^{-1}\frac{\mu}{\sqrt{(1-h^2)}} - \frac{\mu}{\sqrt{(1-h^2-\mu^2)}}\right\}.$$

This, then, is the equation to the curve; if $\omega_{,}$, $\omega_{,,}$ are the extreme values of ω, and $\mu_{,}$, $\mu_{,,}$ those of μ, h can be found from the expression

for $\omega_{,,} - \omega_{,}$. If ε is the angle made by the curve with the meridian at the commencement, then $h = \cos\theta_{,} \sin\varepsilon\,(1 + e_1 \sin^2\theta_{,})$, and as an approximate value of ε, we have

$$\cot\varepsilon = \frac{\tan\theta_{,,}\cos\theta_{,}}{\sin(\omega_{,,}-\omega_{,})} - \sin\theta_{,}\cot(\omega_{,,}-\omega_{,}).$$

For the length of the curve,

$$ds = a\left\{\frac{1+e_1\mu^2}{\sqrt{(1-h^2-\mu^2)}} - \frac{h^2 e_1 \mu^2}{(1-h^2-\mu^2)^{\frac{3}{2}}}\right\} d\mu,$$

$$(32)\quad s = a\left\{\left[1+(1+h^2)\frac{e_1}{2}\right]\sin^{-1}\frac{\mu}{\sqrt{(1-h^2)}} - \frac{e_1}{2}\frac{(1+h^2-\mu^2)\mu}{\sqrt{(1-h^2-\mu^2)}}\right\}.$$

If $h = 0$, this expression gives the length of any arc of a meridian, but in this case,

$$ds^2 = \sqrt{(dr^2 + r^2 d\theta^2)} = r\left\{1 + \frac{e_1^2}{2}\sin^2 2\theta\right\} d\theta,$$

or

$$ds = a\left\{1 + e_1\sin^2\theta + e_2\sin^4\theta + \frac{e_1^2}{2}\sin^2 2\theta\right\} d\theta,$$

$$(33)\quad s = a\left\{\left(1+\frac{e_1}{2}+\frac{e_1^2}{4}+\frac{3}{8}e_2\right)\theta - \frac{e_1+e_2}{4}\sin 2\theta + \frac{2e_1^2-e_2}{32}\sin 4\theta\right\}.$$

19. All areas on the earth's surface bounded by lines, whose equation is (31), can be divided into a finite number of parts, each contained by an arc of a meridian, an arc of a parallel of latitude, and a line whose equation is (31). Let A denote the area of this, then ds, being the element of the meridian, $A = \int \omega r \cos\theta\, ds$, or, neglecting quantities of the second order, $= \int \omega r^2 d\mu = ar\mu\omega - a\int r\mu\, d\omega$, finally putting for $d\omega$ its value, and integrating: —

$$(34)\quad A = ar\mu\omega + \frac{a^2 h}{2}\left\{\frac{1+e_1}{h}\tan^{-1}\frac{\sqrt{(1-h^2-\mu^2)}}{h} + \frac{(1-h^2)e_1}{\sqrt{(1-h^2-\mu^2)}}\right\}.$$

20. Let F denote the force of gravity at the surface, then

$$F = \frac{dV}{dr}\sqrt{1+\left(\frac{dr}{rd\theta}\right)^2} = \frac{dV}{dr}\{1 + 18\,e_1^2\mu^2(1-\mu^2)\}.$$

If A_0, A_1, A_2 are the numerical values of the members of equations (23), then

$$F = (A_0 + A_1 + A_2)\left\{1 - \frac{3A_1 + 10A_2 - 18V_0'c_1^2}{A_0 + A_1}\mu^2 + \frac{\frac{35}{3}A_2 - 18V_0'c_1^2}{A_0}\mu^4\right\}.$$

$$(35)\quad = F_0\{1 + w_1\mu^2 + w_2\mu^4 + \&c.\ldots\}.$$

21. Thus far the general theory of the subject. We shall now assume some particular law of density. Suppose that the matter of which the earth is composed is compressible inversely as its density. This gives $d\varrho = \frac{m^2}{4\pi}\frac{dp}{\varrho}$, m being a constant. Substituting for dp its value, $\varrho\, dV$, and integrating $\varrho = \frac{m^2}{4\pi}V$, no constant is added, because it may be supposed contained in V. Then

$$f(V) = m^2 V - \frac{8\pi^2}{T^2},$$

and from (15) generally $T_i = m^2 r^2 V_i$, but

$$T_0 = m^2 r^2 \left(V_0 - \frac{8\pi^2}{m^2 T^2}\right).$$

Thus (11) becomes

$$(36)\qquad \frac{d.r^2\frac{dV_i}{dr}}{dr} - \{2i(2i+1) - m^2r^2\}V_i = 0.$$

In integrating, the part of V_i involving negative powers of r may be neglected, since it belongs to $\breve{V}$. If a_i is an arbitrary constant, and f^{2i} denotes the operation $\frac{d.}{dr}\frac{1}{r}$ performed $2i$ times, the integral of (36) is $V_i = a_i r^{2i-1} f^{2i}(\sin mr)$. The integral may also be obtained thus: put (36) under this form,—

$$(37)\qquad \frac{d^2.rV_i}{dr^2} + \left\{m^2 - \frac{2i(2i+1)}{r^2}\right\}rV_i = 0.$$

Assume $rV_i = P\sin mr + P'\cos mr$; which by substitution gives

$$\frac{d^2.(P \pm P')}{dr^2} \pm 2m\frac{d.(P \mp P')}{dr} - \frac{2i(2i+1)}{r^2}(P \pm P') = 0.$$

Make $P \pm P' = \beta_0 \pm \beta_1 r^{-1} + \beta_2 r^{-2} \pm \beta_3 r^{-3} +$ &c. ...; then this equation results,

$$\pm 2m(n+1)\beta_{n+1} + \{n(n+1) - 2i(2i+1)\}\beta_n = 0;$$

whence $$\beta_{n+1} = \mp \frac{(n-2i)(n+2i+1)}{(n+1)\,2m}\beta_n,$$

the upper sign being taken when n is even, the lower when odd. Then making $\beta_0 = \pm m^{2i} a_i$, in order to agree with the expression $V_i = a_i r^{2i-1} f^{2i}(\sin mr)$, we have

$$(38) \qquad \beta_n = \pm \frac{(2i-n+1)\ldots(2i+n)}{1.2\ldots n.2^n} m^{2i-n} a_i;$$

the upper sign having place when $2i-n$ is of the forms $4\nu+2$, $4\nu+3$, the lower when it is of the forms 4ν, $4\nu+1$.

$$(39) \quad V = \frac{8\pi^2}{m^2 T^2} + a_0 r^{-1} f^0(\sin mr) + a_1 r f^2(\sin mr)\{1 - 3\mu^2\} + a_2 r^3 f^4(\sin mr)\{1 - 10\mu^2 + \tfrac{35}{3}\mu^4\} + \text{&c.}\ldots;$$

or, expanding $f^{2i}(\sin mr)$ by using (38),

$$(40) \quad V = \frac{8\pi^2}{m^2 T^2} + a_0 \frac{\sin mr}{r} + a_1\left\{\left(\frac{3}{r^2} - m^2\right)\frac{\sin mr}{r} - \frac{3m}{r}\frac{\cos mr}{r}\right\}\{1 - 3\mu^2\} + a_2\left\{\left(\frac{105}{r^4} - \frac{45m^2}{r^2} + m^4\right)\frac{\sin mr}{r} - \left(\frac{105m}{r^3} - \frac{10m^3}{r}\right)\frac{\cos mr}{r}\right\}\{1 - 10\mu^2 + \tfrac{35}{3}\mu^4\} + \text{&c.}\ldots$$

22. Since V contains a constant, the term $\frac{8\pi^2}{m^2 T^2}$ may be neglected, except in finding the value of the density. Moreover, for simplicity, let $a_0 = 1$, and $\frac{V_0'}{m^2 V_0} = -H$; then from (36) and (40) we derive

$$(41) \quad \begin{aligned} V_0'' &= (2H - 1)m^2 V_0, \\ V_0''' &= (2 + (m^2 - 6)H)m^2 V_0, \\ V_1 &= a_1(3H - 1)m^2 V_0, \end{aligned}$$

$$V_1' = a_1 (3 + (m^2 - 9) H) m^2 V_0,$$
$$V_1'' = a_1 (m^2 - 12 - (5 m^2 - 36) H) m^2 V_0,$$
$$V_2 = a_2 (m^2 - 35 - (10 m^2 - 105) H) m^2 V_0,$$
$$V_2' = a_2 (-10 m^2 + 175 - (m^4 - 65 m^2 + 525) H) m^2 V_0.$$

By substituting these expressions in (21) and (25), neglecting quantities of the second order, they become, after removing the factor $m^2 V_0$,

$$(3H - 1) a_1 - H c_1 = 0,$$

and

$$\{3 + (m^2 - 9) H\} a_1 + (3H - 1) c_1 - \tfrac{5}{6} q H = 0;$$

whence

$$(42) \quad a_1 = \tfrac{5}{6} q \frac{H^2}{m^2 H^2 - (3H - 1)}, \qquad c_1 = \tfrac{5}{6} q \frac{H(3H - 1)}{m^2 H^2 - (3H - 1)}.$$

23. Represent the volume of the earth by v, its superficial density by R, its mean density by R'. Then true to quantities of the first order, $v = \frac{4\pi}{3}$; and if (V) denote the value of V at the surface $R = \frac{m^2}{4\pi}(V)$. Since b_0 is the mass of the earth, $R' = \frac{b_0}{v} = \frac{3 b_0}{4\pi}$. Hence $\frac{m^2 (V)}{b_0} = \frac{3R}{R'}$, or, putting for b_0 its value from (24),

$$\frac{V_0'}{m^2 (V)} = - \frac{R'}{3R (1 + \frac{2}{3} q)}.$$

If in the expressions for V we make $3\mu^2 = 1$, and consequently $r = 1$,

$$(V) = \frac{8\pi^2}{m^2 T^2} + V_0 = - \frac{2q}{m^2} V_0', \quad \text{then} \quad \frac{H}{1 + 2qH} = \frac{R'}{3R (1 + \frac{2}{3} q)}, \quad \text{or}$$

$$(43) \qquad H = \frac{R'}{3R - 2q (R' - R)}.$$

To find m we have, since $H = - \frac{V_0'}{m^2 V_0}$,

$$(44) \qquad \frac{1}{m} \left(\frac{1}{m} - \cot m \right) = H.$$

24. In order to test the preceding theory by numerical calculation, we adopt the following values for q, R, R', the best we can find: —

$$q = \tfrac{1}{288}, \qquad R = 2.55, \qquad R' = 5.67.$$

We shall mark with an accent the numbers of the formulas from which the numerical values of the following quantities are obtained: —

$$(43)' \quad H = 0.739083,$$

$$(44)' \quad \begin{cases} m &= 146^\circ\ 12'\ 19''.8, \\ &= 2.551765, \end{cases}$$

$$(42)' \quad \begin{cases} a_1 &= 0.00067556, \\ c_1 &= 0.0011126, \end{cases}$$

$$(41)' \quad V_0'' = 0.478166\, m^2 V_0,$$

$$(41)' \quad \begin{cases} V_0''' = 2.378\, m^2 V_0, \\ V_1 = 1.217249\, m^2 V_0 a_1, \\ V_1' = 1.160794\, m^2 V_0 a_1, \\ V_1'' = -2.944\, m^2 V_0 a_1, \\ V_2 = 0.990\, m^2 V_0 a_2, \\ V_2' = 3.345\, m^2 V_0 a_2. \end{cases}$$

To obtain the values of a_1, c_1 true to quantities of the order of q^2, by substituting the preceding in (21) and (25), we have

$$(21)' \quad 0.739083\, c_1 - 1.217249\, a_1 + 0.000000666 = 0,$$

$$(25)' \quad 1.217249\, c_1 + 1.160794\, a_1 - 0.002141513 = 0.$$

$$(21)' \quad 0.739\, c_2 - 0.990\, a_2 - 0.000000900 = 0,$$

$$(25)' \quad 2.695\, c_2 + 3.345\, a_2 - 0.000005243 = 0.$$

$$a_1 = 0.000676848, \qquad c_1 = 0.001113848,$$

$$a_2 = 0.000000283, \qquad c_2 = 0.000001595.$$

$$(18)' \text{ \& } (24)' \quad \breve{V} = b_0 \{ r^{-1} + 0.0005430\, r^{-3}(1 - 3\mu^2) + 0.0000010\, r^{-5}(1 - 10\mu^2 + \tfrac{35}{3}\mu^4) \}.$$

$$(27)' \quad r = a\{1 - 0.003353781 \sin^2\theta + 0.000018608 \sin^4\theta\}.$$

$$(28)' \quad e = 0.003335173 = \frac{1}{299.834}.$$

$$(29)' \quad \theta' = \theta + 679''.09 \sin 2\theta - 1''.34 \sin^4\theta.$$

$$(30)' \quad \theta = \theta' - 679''.09 \sin 2\theta' + 3''.66 \sin^4\theta'.$$

$$(33)' \quad s = a\{0.9983329\,\theta + 0.0008338 \sin 2\theta + 0.0000001 \sin 4\theta\}.$$

$$(35)' \quad F = F_0\{1 + 0.0054093 \sin^2\theta - 0.0000539 \sin^4\theta\}.$$

The following table contains the values of ϱ and of e, the compression of the surfaces of level, for every tenth of the equatorial radius, calculated from the equations

$$\varrho = \frac{m^2}{4\pi} V = \frac{R \sin mr}{r \sin m}, \quad \text{and} \quad e = -\frac{3 V_1}{V_0'} = 3 a_1 \left\{ \frac{3}{r} - \frac{m}{\frac{1}{mr} - \cot mr} \right\}.$$

$\frac{r}{a}$	ϱ	e	$\frac{r}{a}$	ϱ	e
0.0	11.699	0000	0.6	7.635	1–586th
0.1	11.572	1–3675th	0.7	6.398	1–488th
0.2	11.197	1–1872th	0.8	5.108	1–412th
0.3	10.589	1–1240th	0.9	3.810	1–349th
0.4	9.770	1–915th	1.0	2.550	1–300th
0.5	8.773	1–720th			

NOTES AND QUERIES.

I. A Logical Outline of Arithmetic. — One great error in our systems of elementary instruction is the lack of logical method. With the desire of lending a word toward the correction of this error, we present the following logical outline of Arithmetic.

All numerical ideas begin with the unit. It is the origin, the basis of Arithmetic. The unit can be increased and divided, hence arise Numbers and Fractions. Every number is a *synthesis*, every fraction the result of an *analysis*. When by synthesis we have obtained a number, by analysis we may pass to a smaller number; hence numbers can be increased and diminished, and these are the only operations to which they may be subjected. All other processes are merely modifications of, or are based upon, these fundamental ones.

To determine *when* and *how* to increase, and *when* and *how* to diminish, we employ a process of reasoning called Comparison. This

reasoning process also gives rise to several particular arithmetical processes. Arithmetic of whole numbers — and the same obtains with fractions — is therefore reduced to the two general processes of Synthesis and Analysis, these to be controlled by the logic of Comparison. Let us glance briefly at each.

A general synthesis is Addition. Multiplication is a special case of addition, in which the numbers to be added are equal, — the sum in this case being distinguished by the term *product*. Composite numbers, formed by the synthesis of factors; Multiples formed by the synthesis of *particular* factors, and Involution, requiring a product of *equal* factors, are all included under Multiplication.

A general analysis is called Subtraction. Division is a *special* analysis, in which equal numbers are successively subtracted, with the additional object of ascertaining how many times such subtractions may be performed. Factoring is a special case of division, in which many or all of the divisors of a member are required; Evolution, a special case of factoring, in which *one* of several *equal* factors is required; and Common Divisor, a case of factoring in which some common factor of several numbers is required.

In comparing numbers we observe we may pass from one to another of different species under the same genus, and thus we have Reduction. By Comparison we obtain Ratio, arithmetical and geometrical, from the first of which, by a farther comparison, arises Arithmetical Progression; and from the second we obtain, in a similar manner, Proportion and Geometrical Progression.

Thus we derive a complete outline of the science of numbers. The rest of Arithmetic consists of the solution of problems, either real or theoretic, and may be included under the head of Applications of Numbers. Arithmetic is therefore pure and applied, or abstract and concrete. The following is our logical outline of abstract Arithmetic: —

SYNTHESIS.	Addition.	{Multiplication.	Forming Composite Numbers. Involution. Multiples.
ANALYSIS.	Subtraction.	{Division.	Factoring. Evolution. Common Divisor.
COMPARISON.			Reduction. Ratio. Proportion. Progressions.

It is believed that the above view of Arithmetic must tend to simplify the subject, and that much clearer notions of the science will be obtained when these philosophical relations are understood. — EDWARD BROOKS, Prof. of Math., State Normal School, Millersville, Pa.

II. *Note on finding the expressions for the sines and cosines of the sum and difference of two angles in terms of the sines and cosines of the angles themselves, by using the equations for passing from a system of rectangular co-ordinates to another system also rectangular.*

These equations are as follows: —

(1) $$y = x' \sin a + y' \cos a,$$

(2) $$x = x' \cos a - y' \sin a.$$

But $y = \sin(\alpha + \beta)$, and $x = \cos(\alpha + \beta)$.

Also $y' = \sin \beta$, and $x' = \cos \beta$.

Substituting these values in (1) and (2), we have

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \sin \beta \cos \alpha,$$

and $$\cos(\alpha + \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta.$$

If the point P falls between the axes of x, then y' is negative, and the equations become

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \sin\beta\cos\alpha,$$

and $$\cos(\alpha - \beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

— M. C. STEVENS, Professor of Mathematics, Haverford College, Pa.

III. The following similarity in the form of equations may not be uninteresting to some readers of the Monthly.

$\frac{x}{a} + \frac{y}{b} = 1$ is the equation of the straight line,

$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = 1$ is the equation of ellipse or hyperbola,

according as we take the upper or lower sign, the origin being the centre.

$\frac{xx'}{a^2} \pm \frac{yy'}{b^2} = 1$ is the equation to the tangent line to the ellipse or hyperbola.

In these equations, a is the intercept on the axis of x, and b is the intercept on the axis of y.

In the case of the hyperbola, the intercept on the axis of y is imaginary.

These forms, by their similarity, are easily remembered. I take them from SALMON'S Conic Sections. — M. C. S.

IV. *Notes on Geometry.* — We think that nothing ought to be admitted in Geometry as self-evident which admits of proof. The only basis of geometrical reasoning ought to be the definitions, and those intuitive truths which can be reduced to no simpler elements, and which it is impossible to deny.

Now it can be *proved* that all right angles are equal, and LEGENDRE has given the proof satisfactorily; why, then, is this truth taken for granted in most of our Geometries? It does not follow that because it is sufficiently obvious to some minds, it is so to all.

Again, in order that any assumed measure of the inclination of two planes may be a just measure, it is necessary that the measure shall be equal for equal inclinations, that it shall vary as the inclinations vary, and that it shall be the same at whatever point of the intersection of the two planes it may be applied; and yet, with all these conditions to be satisfied, it is taken for granted that the angle contained between two perpendiculars, one in each plane and perpendicular to the common intersection at the same point, is a just measure of the inclination of the planes. It is, to say the least, quite as self-evident that an angle is measured by the arc intercepted between its sides and described from its vertex as a centre. Then why demonstrate this last?

It is more excusable to assume, that only one line can be drawn through a given line parallel to a given line, because it was very difficult to prove this proposition in an elementary way, and very difficult to establish the theory of parallel lines without it.

The following method seems elementary enough and entirely satisfactory. Let ab and de be two parallels, and ae a line making any angle with ab. Now however small the angle bac, it may be made, by being repeated a definite number of times around a, to fill up the whole plane space about a; it is, therefore, when the sides ac and ab are indefinitely produced, a finite part of that space. But the space between the parallels ab and de, extended indefinitely, can never, by any number of repetitions, be made to fill up the whole plane space, and it must therefore be an infinitely small part of that space. Hence, if ab and ac be sufficiently extended, the space between ab and ac must ultimately become greater than the space between ab and de, which requires that ac shall intersect de, and pass beyond it.

This method, we believe, was first suggested by one of the Bernouillis, and we see no objection to it, except that the idea of infinity

is involved in the reasoning; and yet this same idea meets us in the very definition of parallel lines in the theory of the circle, and in incommensurable quantities.

The mention of incommensurables brings us to our last remark: it is, that LEGENDRE'S method of treating these quantities leaves the learner with very vague and indistinct ideas of them. We have two angles, A and B, and their arcs C and D, and we prove the relation $\frac{A}{B} = \frac{C}{D}$. If the learner asks what $\frac{A}{B}$ and $\frac{C}{D}$ mean, we tell him that all ratios are essentially and necessarily *numbers*, and cannot be anything else; but if A and B are incommensurable, $\frac{A}{B}$ cannot be expressed by any conceivable number, nor can $\frac{C}{D}$; we may say that whatever expresses $\frac{A}{B}$ will also express $\frac{C}{D}$; but he may reply that no number can be made to express either, and how shall we get out of the difficulty? Simply by showing from the outset that incommensurable quantities approximate to commensurable as their divisor is made smaller and smaller, and that as the relation $\frac{A}{B} = \frac{C}{D}$ subsists at each step in the approximation, it will continue to be true, however far the process be carried; and yet as $\frac{A}{B}$ and $\frac{C}{D}$ are at each step ratios between commensurable quantities, they are always actual numbers.

To illustrate: suppose $A = 3$ and $B = 5 + R$, then $C = 3$ and $D = 5 + r$; neglecting R and r $\frac{A}{B} = \frac{3}{5} = \frac{C}{D}$. Now take a divisor of A and B smaller than R; C and D will be divided into parts smaller than r; and let

$$A = 7 \text{ and } B = 12 + R', \qquad C = 7 \text{ and } D = 12 + r'.$$

Then neglecting R' and r' $\frac{A}{B} = \frac{7}{12} = \frac{C}{D}$. And so on. — PIKE POWERS, Staunton, Va.

V. *Problem.* — In any triangle, plane or spherical, if α, β, γ denote the perpendiculars dropped from the angles upon the opposite sides,

ϱ, r the radii of the inscribed and circumscribed circles, δ the distance between the centres; $\varrho', \varrho'', \varrho'''$ the radii of the escribed circles, $\delta', \delta'', \delta'''$ the distances between the centres of these circles and the centre of the circle whose radius is r, prove that in the plane

(1) $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{\varrho}$, (2) $-\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{\varrho'}$,

(3) $\frac{1}{\alpha}-\frac{1}{\beta}+\frac{1}{\gamma}=\frac{1}{\varrho''}$, (4) $\frac{1}{\alpha}+\frac{1}{\beta}-\frac{1}{\gamma}=\frac{1}{\varrho'''}$.

Squaring these four equations, and adding, we get

(5) $$\frac{4}{\alpha^2}+\frac{4}{\beta^2}+\frac{4}{\gamma^2}=\frac{1}{\varrho^2}+\frac{1}{\varrho'^2}+\frac{1}{\varrho''^2}+\frac{1}{\varrho'''^2},$$

Also, as is easily seen,

(6) $$\frac{8}{\alpha\beta}=\frac{1}{\varrho^2}+\frac{1}{\varrho'''^2}-\frac{1}{\varrho'^2}-\frac{1}{\varrho''^2}, \quad \frac{8}{\alpha\gamma}=\frac{1}{\varrho^2}+\frac{1}{\varrho''^2}-\frac{1}{\varrho'^2}-\frac{1}{\varrho'''^2},$$
$$\frac{8}{\beta\gamma}=\frac{1}{\varrho^2}+\frac{1}{\varrho'^2}-\frac{1}{\varrho''^2}-\frac{1}{\varrho'''^2}.$$

In the sphere,

(1) $$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}=\frac{1}{\sin\varrho}\,\frac{\cos\delta}{\cos r},$$

(2) $$-\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}=\frac{1}{\sin\varrho'}\,\frac{\cos\delta'}{\cos r},$$

(3) $$\frac{1}{\sin\alpha}-\frac{1}{\sin\beta}+\frac{1}{\sin\gamma}=\frac{1}{\sin\varrho''}\,\frac{\cos\delta''}{\cos r},$$

(4) $$\frac{1}{\sin\alpha}+\frac{1}{\sin\beta}-\frac{1}{\sin\gamma}=\frac{1}{\sin\varrho'''}\,\frac{\cos\delta'''}{\cos r}.$$

Equations corresponding to (5), (6) in the plane can be found in the same way. Analogies (2), (3), (4) are new to me, but it is hardly probable that so simple relations have not before been observed. — David Trowbridge, Perry City, N. Y.

VI. *Magic Table.* — The following curious and ingenious table is taken from an old newspaper, where it appears under the head of "*Magic Table to tell a Lady's Age.*"

1st.		2d.		3d.		4th.		5th.		6th.	
1		2		4		8		16		32	
1	33	2	34	4	36	8	40	16	48	32	48
3	35	3	35	5	37	9	41	17	49	33	49
5	37	6	38	6	38	10	42	18	50	34	50
7	39	7	39	7	39	11	43	19	51	35	51
9	41	10	42	12	44	12	44	20	52	36	52
11	43	11	43	13	45	13	45	21	53	37	53
13	45	14	46	14	46	14	46	22	54	38	54
15	47	15	47	15	47	15	47	23	55	39	55
17	49	18	50	20	52	24	56	24	56	40	56
19	51	19	51	21	53	25	57	25	57	41	57
21	53	22	54	22	54	26	58	26	58	42	58
23	55	23	55	23	55	27	59	27	59	43	59
25	57	26	58	28	60	28	60	28	60	44	60
27	59	27	59	29	61	29	61	29	61	45	61
29	61	30	62	30	62	30	62	30	62	46	62
31	63	31	63	31	63	31	63	31	63	47	63

The peculiarity of the table is this: If all the columns in which a particular number is found be indicated, the sum of the numbers standing at the head of such columns will equal the particular number. For instance, a particular number is found in the 1st, 2d, 4th, 5th, and 6th columns; what is the number? The sum $1 + 2 + 8 + 16 + 32 = 59$ is the required number. — Artemas Martin, Franklin, Venango County, Pa.

VII. In Robinson's Algebra is found the following problem: — "A man borrowed $60; what sum shall he pay daily to cancel the debt, principal and interest, in 60 days; interest at 10 per cent for 12 months of 30 days each?" $\$1\frac{5}{60}$ and $\$1\frac{61}{720}$ are the answers given in different editions. Is either of them *correct*, or is the true answer $\$1\frac{61}{720}$? — D. W. Hoyt, Professor of Mathematics, New Hampton Institute, Fairfax, Vt.

Mathematical Monthly Notices.

Annual Report of the Board of Regents of the Smithsonian Institution, showing the Operations, Expenditures, and Condition of the Institution for the Year 1861.

This Report opens, as usual, with that of PROFESSOR HENRY, — the Secretary of the Institution, — giving the history of its transactions for the year; from which we learn that the eleventh volume of the "Smithsonian Contributions to Knowledge" is nearly ready, containing the following memoirs: On North American Oölogy, by DR. THOMAS M. BREWER; On the Total Eclipse of the Sun, September 7, 1858, observed near Olmos, Peru, by LIEUT. GILLIS; Discussion of the Magnetic and Meteorological Observations made at the Girard College Observatory, Philadelphia, from 1840 to 1845, by A. D. BACHE, LL. D.; A Memoir, by DR. JOHN LE CONTE, on the Coleoptera, or Beetles, known to inhabit the Middle and Eastern portions of the great Central Regions of temperate North America; Result of Magnetical and Hypsometrical Observations in Mexico, — to which is appended Notes on the Volcano of Popocatapetl and its vicinity; On the American Storm of December 20, 1836, and the European Storm of December 25, of the same year, by PROFESSOR ELIAS LOOMIS. The General Appendix of the Report contains: Four Lectures on Agricultural Chemistry, by PROFESSOR S. W. JOHNSON; Lectures on the Shells of the Gulf of California, by P. P. CARPENTER; Latest Researches of MAEDLER relating to the General Movement of the Stars around a Central Body; Report on the Transactions of the Society of Physics and Natural History of Geneva, from July, 1858, to June, 1859; Present State of Ethnology in relation to the form of the Human Skull; Memoir of PYRAMUS DE CANDOLLE; On the Means which will be available for Correcting the Measure of the Sun's Distance in the next twenty-five years; Reports on the State of Knowledge of Radiant Heat, made to the British Association for the Advancement of Science, at the Meetings in 1832, 1840, and 1854; Description of the Magnetic Observatory at the Smithsonian Institution; On the use of the Galvanometer as a Measuring Instrument; On Observation of Earthquake Phenomena; Meteorological Instruments; On filling Barometer Tubes; Account of the Construction of a Standard Barometer, and Description of the Apparatus and Processes employed in the Verification of Barometers at the Kew Observatory.

The following problems are from "The Lady's and Gentleman's Diary" for the year 1861: —

Question I. Given the sum of the two sides, the radius of the escribed circle touching the base externally, and either of the angles at the base, to construct the triangle. W. H. LEVY, Shalbourne.

II. If the distance between the angular points and the points of contact of the escribed circles of a triangle be in harmonical progression, the escribed radii will be in arithmetical progression, and *vice versa*. — THOMAS WILKINSON, Burnley.

III. If, from the angular points of the triangle ABC, two lines be drawn, making the same constant angles with the adjacent side, two triangles $A'B'C$, $A''B''C''$ are formed similar to the original; and the circles $AA'A''$, $BB'B''$, $CC'C''$ pass through the same point. Also, if r_1, r_2, r_3 be the radii of these circles, and R that of ABC; then $r_1, r_2, r_3 = R_2$: and if O_1, O_2, O_3, O be the centres of the circles, $16 \Delta ABC \times \Delta O_1, O_2, O_3 = a^2b^2 + b^2c^2 + c^2a^2$ — EDWARD RUTTER, Sunderland.

IV. Given the perpendicular, the vertical angle and the sum of the three sides to construct the triangle. — W. H. LEVY, Shalbourne.

V. From a small plane on the floor of a room the horizontal breadth of a window in a, and the angles of elevation of its top and bottom are β and β' respectively. Show that the direct illumination of the plane by the light of the sky through the window, is to its illumination by the whole sky, supposed of uniform brightness, as $x \sin(\beta + \beta') \sin(\beta - \beta')$ to 2π. — THOMAS DOBSON, B. A., Liverpool.

VI. PC, PC', at right angles to each other, are tangents to an ellipse, and $CQ, C'Q$ the corresponding normals. Show that the locus of the intersection of CC', PQ bisects the ellipse. — STEPHEN WATSON, Haydonbridge.

VII. Let ABC be any triangle; O, O_1, O_2, O_3 the centres of its inscribed and escribed circles respectively. Then the twelve circles described about the component triangles of the complete quadrilaterals $O_1O_2OO_3$, $O_2O_1OO_3$, $O_1O_2OO_1$, will intersect, four and four, in A, B, C and their centres will lie, two and two, in six points on the circumference of the circumscribing circle. — T. T. WILKINSON, Burnley.

VIII. A circular target, for rifle-practice, weighing 18lbs., and measuring 2 feet in diameter is suspended from a horizontal axis, about which it is capable of revolving freely, the centre of the target being at a distance of 6 feet from the axis. Find at what distance from the axis it must be struck by a rifle ball, weighing 8 ounces, and moving with a given velocity, in order that the ball and target may move off together with the greatest possible velocity. — STEPHEN FENWICK, of the Royal Military Academy, Woolwich.

IX. O_1, O_3 are the centres of the escribed circles, touching the sides AC, AB, of the triangle ABC, externally. Join O_1B, O_3C, cutting the sides in P_1, Q_1; O_1Q_1, O_3P_1, cutting them in P_2, Q_2, and so on; then the distance of any point in the line P_nQ_n, from BC, is equal to n times the sum of its distances from AC, AB. Also the lines $P_1Q_1, P_2Q_2, \ldots\ldots P_nQ_n$, all pass through the same fixed point. — STEPHEN WATSON, Haydonbridge.

X. If from the angles of a triangle Δ, straight lines be drawn to the opposite sides, dividing them in the proportion of m to n, where $m + n = 1$, and a triangle Δ_1 be formed having the dividing lines as sides, show that $\Delta_1 = \Delta(1 - mn)$. — PETRARCH.

XI. If a straight line be drawn parallel to a given straight line, and on the same side of it with a given circle; generally four circles may be described through any point in the parallel to touch the given straight line and to touch the circle, two externally and two internally. The ratio of a mean proportional betwixt the radii of external contact to a mean proportional betwixt the radii of internal contact, will be the same, wheresoever on the parallel the point be taken; and if the parallel be drawn through the centre of the circle, it will be a ratio of equality. Required a geometrical proof. — REV. WM. MASON, Normanton, Yorkshire.

XII. Compare the times of the small oscillations of a rod of any given length, not exceeding the earth's diameter; 1st, when it is suspended by one extremity, the other just touching the earth's surface; 2dly, when it is suspended by one end at the surface, and vibrates in a diametrical perforation through the earth. — C. H. BROOKS, Great George Street, Westminster.

XIII. A small horizontal orifice is made at the bottom of a vessel containing fluid. Determine the shape of the vessel, so that the form of the jet, at the instant the vessel is empty, shall be a straight line. — REV. S. TEBAY, Rivington.

XIV. Three points being taken at random in space as the corners of a plane triangle, determine the probability that it shall be acute. — THE EDITOR.

XV. If an ellipse be inscribed in an equilateral triangle so as to have one of its foci on the inscribed circle, then the other focus will lie on the hypocycloid of three branches, whose cusps are at the angular points of the triangle. — "CONIC," of St. John's College, Cambridge.

Editorial Items.

Mr. G. W. Hill, the author of the Prize Essay printed in this number of the Monthly, is a graduate of Rutgers College, New Brunswick, N. J., and at present a member of the Lawrence Scientific School, Cambridge, Mass., pursuing his Mathematical studies under the direction of Professor Peirce.

Sir John Herschel's Portrait. — In our next number we shall present to our readers a Steel Portrait of Sir John Herschel, engraved by H. Wright Smith, Esq., of Boston, from a recent photograph, which is pronounced an excellent likeness. The number will also contain a complete catalogue of the titles of the distinguished author's works, in all *two hundred and twenty-four*.

The following gentlemen sent solutions of Prize Problems in the October number: John A. Weinbrener, Princeton College, N. J.; Asher B. Evans, Madison University, Hamilton, N. Y.; E. O. Gibson, South New Berlin, N. Y.; J. O. Ellinwood, and D. Martin, Liberal Institute, Clinton, N. Y.; Thomas M. Sechler, E. K. Leonard, Eugene B. Willard, Marietta College, Ohio; G. A. Osborne, Jr., Lawrence Scientific School; G. W. Lewis, Middlebury College, Vt.; William Timpson, White Plains, N. Y.; A. G. Barker, Waterville College, Me.; H. Teiman, Baltimore, Md.; Joseph H. Buckley, Yale Scientific School; G. B. Hicks, Cleveland, Ohio; John F. Curtis, Piedmont Institute, Liberty, Va.

Solutions of the Problems in the November number have been received from the following gentlemen: Edwin A. Dodd, Gouverneur, N. Y.; W. D. Mershon, Princeton College, N. J.; David Trowbridge, Perry City, N. Y.; G. A. Osborne, Jr., Lawrence Scientific School; G. B. Hicks, Cleveland, Ohio; Asher B. Evans, Hamilton College, N. Y.; A. G. Barker, Waterville College, Me.; H. Teiman, Baltimore, Md.; William Woolsey Johnson, Yale College, New Haven, Ct.

Books Received. — A Higher Arithmetic: embracing the Science of Numbers, and the Art of their Application. By A. Schuyler, M. A., Principal of Seneca C. Academy, and Professor (elect) of Mathematics in Baldwin University. New York: Sheldon & Co., 115 Nassau street. 1860. — Plane Trigonometry, for the Use of Colleges and Schools. With Numerous Examples. By I. Todhunter, M. A., Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge. Second edition, revised. Cambridge: MacMillan & Co. 1860. — The Claim of Leibnitz to the Invention of the Differential Calculus. By Dr. H. Sloman. Translated from the German, with Considerable Alterations and New Addenda by the Author. Cambridge: MacMillan & Co. — Dodd's Mathematical Text-books. Elementary and High School Arithmetics, 2 vols.; Elementary and High School Algebras, 2 vols.; Elements of Trigonometry, Plane and Spherical; with Applications to Heights and Distances, the Areas of Polygons, Surveying, Navigation, and the Solution of Astronomical Problems; Elements of Geometry and Mensuration. New York: Published by Pratt, Oakley, & Co.

. Annual Report of the Board of Regents of the Smithsonian Institution, showing the Operations, Expenditures, and Condition of the Institution for the year 1859. Washington. 1860. — The Quarterly Journal of Pure and Applied Mathematics. No. 14. October, 1860. London: John W. Parker & Son. — Nouvelles Annales de Mathématiques. Tome Vingtième. Augmenté d'un Bulletin de Bibliographie, d'Histoire et de Biographie Mathématiques. Janvier, 1861. Paris: Mallet-Bachelier.

J. F. W. Herschel

THE

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MATHEMATICAL MONTHLY.

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PRIZE PROBLEMS FOR STUDENTS.

I. Let ACB, ADB be two right-angled triangles having a common hypothenuse, AB; join CD, and on CD produced both ways draw perpendiculars AE, BF. Show that

$$CE^2 + CF^2 = DE^2 + DF^2.$$

II. A clock gains $3\frac{1}{4}$ minutes in 15 seconds under the 24 hours. At noon it is two minutes slow; when will it indicate correct time?

III. Two railroads intersect at an angle φ; and on these roads, distant from the junction a and b miles respectively, are two trains running towards the junction at the rates of m and n miles respectively. When and where will these trains be at a minimum distance from each other? — Communicated by Prof. Wray Beattie, Wesleyan University, Mt. Pleasant, Iowa.

IV. The former of the two similar triangles $A'B'C'$ and ABC is circumscribed about the triangle DEF, and the latter is inscribed in DEF. Show that DEF is a mean proportional between $A'B'C'$ and ABC only when the homologous sides of the latter are parallel; also determine when ABC is a minimum, $A'B'C'$ and DEF being given. — Communicated by Asher B. Evans, Madison University, Hamilton, N. Y.

V. If r be the radius of the sphere inscribed in a tetrahedron formed by three rectangular co-ordinate planes and the plane

$\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$, and d be the distance of this plane from the origin, prove that $\frac{1}{r}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}$.

Solutions of these problems must be received by June 1st, 1861.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS IN No. III., Vol. III.

The first prize is awarded to George B. Hicks, Cleveland, Ohio.

The second prize is awarded to Asher B. Evans, Madison University, Hamilton, N. Y.

The third prize is awarded to David Trowbridge, Perry City, N. Y.

The fourth prize is awarded to W. W. Lemmon, M'Kendree College, Lebanon, Illinois.

Prize Solution of Problem I.

By W. W. Lemmon, M'Kendree College, Lebanon, Ill.

From AC, the diagonal of a square, $ABCD$, cut off AE equal to one fourth of AC, and join BE, DE. Show that the figure $BADE$ equals twice the square on AE.

The triangles ADE and ABE are equal, because they have the same base, AE, and the equal altitudes; and each is equal to one fourth of ADC or ABC. Therefore $4BADE=AB^2$; and since $AC^2=2AB^2$, we have

$$BADE=\tfrac{1}{4}AB^2=\tfrac{1}{8}AC^2=\tfrac{1}{8}(4AE)^2=2AE^2.$$

Mr. William Timpson says: Since the semi-diagonal AF is bisected in E, EG parallel to FB bisects AB in G. The triangle

$$AEG=\tfrac{1}{2}AE^2=GEB;$$

hence $AEB=AE^2$; $\therefore BADE=2AE^2$. Or, complete the square $FEGH$. Draw BE, cutting GH in L, and produce it to meet DA in M. The triangles EGL, BHL being equal, the triangle

$$BFE=BEA=EF^2=AE^2,$$

as before.

Corollary. — The triangles AME, BCE being similar,

$$(EC = 3AE) : AE = EB : EM = (BC = AD) : AM;$$

or $$3 : 1 = EB : EM = AD : AM; \quad \therefore AM = \tfrac{1}{3} AD;$$

and $$4 : 1 = (EB + EM = BM) : EM; \quad \therefore EM = \tfrac{1}{4} BM.$$

Prize Solution of Problem II.

By E. S. Stearns, Chester, Morris Co., N. J.

Two toothed wheels work against each other. Show that, if the number of teeth in one be prime to that in the other, before two teeth, which have once been in contact, come in contact again, every tooth of the one wheel will have been in contact with every tooth of the other.

Let $a > b$, prime to each other, be the number of teeth in the wheels A and B; then, before any two teeth which are together can come together again, each wheel must make a whole number of revolutions; A makes b, and B makes a revolutions. Again, when B makes one revolution, every tooth of B has been in contact with one tooth of A, and in making a revolutions every tooth of B has been in contact with a, or every tooth of A.

Prize Solution of Problem III.

By George B. Hicks, Cleveland, Ohio.

Eliminate φ from the equations

$$x \cos(\varphi + \alpha) + y \sin(\varphi + \alpha) = a \sin 2\varphi,$$
$$y \cos(\varphi + \alpha) - x \sin(\varphi + \alpha) = 2a \cos 2\varphi,$$

showing that

$$(x \sin\alpha - y \cos\alpha)^{\frac{2}{3}} + (y \sin\alpha + x \cos\alpha)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

Develop the left-hand members of these equations, and divide the first by the second; we get

$$\frac{x\cos\alpha + y\sin\alpha + \tan\varphi\,(y\cos\alpha - x\sin\alpha)}{y\cos\alpha - x\sin\alpha - \tan\varphi\,(x\cos\alpha + y\sin\alpha)} = \frac{\tan 2\varphi}{2} = \frac{\tan\varphi}{1 - \tan^2\varphi}.$$

$$\therefore \tan^3\varphi = \frac{x\cos\alpha + y\sin\alpha}{y\cos\alpha - x\sin\alpha} = \frac{m}{n}.$$

$$\therefore \sin\varphi = \frac{m^{\frac{1}{3}}}{(m^{\frac{2}{3}}+n^{\frac{2}{3}})^{\frac{1}{2}}}, \qquad \cos\varphi = \frac{n^{\frac{1}{3}}}{(m^{\frac{2}{3}}+n^{\frac{2}{3}})^{\frac{1}{2}}}.$$

But the first equation may be written

$$\frac{m}{\sin\varphi} + \frac{n}{\cos\varphi} = 2a = (m^{\frac{2}{3}}+n^{\frac{2}{3}})^{\frac{3}{2}}.$$

Hence, observing that $n^{\frac{1}{3}}$ and $(-n^{\frac{1}{3}})$ are identical, we obtain

$$m^{\frac{2}{3}} + n^{\frac{2}{3}} = (x\sin\alpha - y\cos\alpha)^{\frac{2}{3}} + (y\sin\alpha + x\cos\alpha)^{\frac{2}{3}} = (2a)^{\frac{2}{3}}.$$

This elegant solution is the only complete one received, owing doubtless to the misprint of a sign in the problem.

Prize Solution of Problem IV.

By Asher B. Evans, Madison University, Hamilton, N. Y.

Show that $\frac{1}{4}m^2\sqrt{3}$ is the area of the greatest triangle which can be formed with the lines a, b, c, subject to the condition $a^3+b^3+c^3=3m^3$.

If A denotes the area of the triangle, then

$$u = 16A^2 = 2a^2b^2 + 2a^2c^2 + 2b^2c^2 - a^4 - b^4 - c^4$$

must be a maximum subject to the given condition. Therefore

$$(1) \qquad \tfrac{1}{4}du = (ab^2 + ac^2 - a^3)\,da + (ba^2 + bc^2 - b^3)\,db + (ca^2 + cb^2 - c^3)\,dc = 0,$$

$$(2) \qquad a^2\,da + b^2\,db + c^2\,dc = 0.$$

Multiply (2) by λ, an indeterminate multiplier, add the product to (1), and equate the coefficients of the arbitrary differentials da, db, dc, to zero, which gives

$$a(b^2+c^2-a^2+\lambda a) = 0, \qquad b(a^2+c^2-b^2+\lambda b) = 0,$$
$$c(a^2+b^2-c^2+\lambda c) = 0.$$

In these three equations, put $b = na$, $c = ra$, and they become

$$(n^2+r^2-1)a+\lambda=0, \qquad (1+r^2-n^2)a+\lambda n=0,$$
$$(1+n^2-r^2)a+\lambda r=0.$$

By adding the first and second, first and third, and second and third of these equations, we get

$$2r^2a+\lambda(1+n)=0, \quad 2n^2a+\lambda(1+r)=0, \quad 2a+\lambda(r+n)=0;$$

$$\therefore \lambda=-\frac{2r^2a}{1+n}=-\frac{2n^2a}{1+r}=-\frac{2a}{r+n},$$

$$\therefore r^3=n^3+n^2-r^2=n+1-r^2n; \quad \therefore \{(n+1)^2+r^2\}(n-1)=0;$$

$$\therefore n-1=0, \qquad \text{and} \qquad (n+1)^2+r^2=0.$$

Since the sum of the squares of two real quantities cannot be zero, the only real value of n is unity, which gives $r=1$. Therefore $a=b=c$, and from the given condition

$$a=b=c=m; \quad \therefore A=\tfrac{1}{4}m^2\sqrt{3}.$$

Mr. TROWBRIDGE shows that $A=\frac{1}{4}m^2\sqrt{3}$ is also the maximum triangle when $a^n+b^n+c^n=3m^n$.

PRIZE SOLUTION OF PROBLEM V.

The equation of a family of ellipses is $ax^2+by^2=1$, when $a-b=c$, a constant. Show that the curve, which cuts all the individuals at an angle whose tangent is $\frac{y}{x}$, is $x=Ce^{\frac{cy^2}{2}}$.

Let $\theta, \theta_1, \theta_2$ be the angles made by the tangents to the ellipse, and trajectory with each other and with the axis of x. By trigonometry, we have

$$(1) \qquad \tan\theta=\frac{\tan\theta_1-\tan\theta_2}{1+\tan\theta_1\tan\theta_2},$$

since $\tan\theta=\tan(\theta_1-\theta_2)$.

But $\tan\theta_1=\frac{dy}{dx}=-\frac{ax}{by}$ form the equation to the family of

ellipses, and $\tan\theta_2 = \frac{dy}{dx}$ as referred to the trajectory; and since $\tan\theta = \frac{y}{x}$, (1) becomes, by substitution and reduction,

$$(ax^3 + by^2)\,dx - (a-b)\,xy\,dy = 0;$$

$\therefore \frac{dx}{x} - cy\,dy = 0$, or, by integration, $\log x - \frac{cy^2}{2} = \log C$, or, passing to exponentials, $x = Ce^{\frac{cy^2}{2}}$.

This solution was substantially given by all the competitors.

SOLUTION OF NUMERICAL EQUATIONS OF THE THIRD DEGREE.

By Prof. J. C. Porter, Liberal Institute, Clinton, N. Y.

The following method of obtaining the approximate roots of cubic equations is analogous to Horner's, in being based upon the principle of successive transformations. But in obtaining the derived polynomials the common method of synthetical division is departed from, the trial divisors, corrections, and complete divisors being expressed by formulas, which give to the operation a modified form. This modification is supposed to possess the following advantages: —

1. The operation is somewhat abbreviated, even in the absence of decimal contraction.

2. The application of decimal contraction is more perfect, especially in equations having extended decimal coefficients.

3. In the several columns, the terms which correspond to any figure of the root are all situated between the same two horizontal lines, — an arrangement which facilitates a review of the work and the correction of mistakes.

Let us assume, for the general form,

(1) $$y^3 + sy^2 + py = q;$$

and put $y = m + r$, in which m represents the initial figure or figures of the root, and r the remaining part. Substituting this value of y in equation (1), and factoring with reference to r, we have,

$$(2)\quad (m^3 + s m^2 + p m) + (3 m^2 + 2 s m + p) r + (3 m + s + r) r^2 = q.$$

Let
$$\delta = m^3 + s m^2 + p m,$$
$$\delta' = 3 m^2 + 2 s m + p,$$
$$\delta'' = 3 m + s + r;$$

then equation (2) becomes

$$\delta + \delta' r + \delta'' r^2 = q;$$

or,

$$(3)\quad \delta + (\delta' + \delta'' r) r = q.$$

From this equation we perceive that, if δ be subtracted from the absolute term, q, the remainder may be divided by δ', as a trial divisor, to obtain r, the next figure of the root; the divisor may then be completed by the addition of $\delta'' r$. It is evident, also, that the new trial divisor for obtaining r_1, the next figure of the root, will be what δ' will become when $m + r$ is substituted for m. This we find to be

$$(3 m^2 + 2 s m + p) + 2 (3 m + s + r) r + r^2 = \delta' + 2 \delta'' r + r^2$$
$$= \delta' + \delta'' r + (\delta'' r + r^2).$$

To correct this trial divisor we must use, instead of δ'', what δ'' becomes when $m + r$ is substituted for m, and r_1 for r; this we find to be

$$3 m + 3 r + s + r_1 = 3 m + r + s + (2 r + r_1) = \delta'' + (2 r + r_1).$$

To recapitulate, we have,

δ = first subtrahend.

δ' = trial divisor.

δ''	= first factor of correction.
$\delta'' r$	= correction.
$\delta' + \delta'' r$	= complete divisor.
$\delta' + \delta'' r + \delta'' r + r^2$	= new trial divisor.
$\delta'' + 2r + r_1$	= first factor of new correction.
$(\delta'' + 2r + r_1) r_1$	= new correction.

It will be seen by inspecting these formulas, —

1st. That the new trial divisor is formed by adding to the last complete divisor the last correction and the square of the last figure of the root.

2d. The first factor of the new correction is formed by adding to the first factor of the former correction twice the former figure of the root plus the new figure. In all cases *algebraical* addition is understood.

Let it be required to find one value of y in the equation

$$y^3 - 4y^2 + 5y = 12.$$

We find, by trial, the initial figure, $3 = m$; whence we obtain $\delta = 6$, and $q - \delta = 6.000$, the first dividend. We next obtain the trial divisor, $\delta' = 8.00$, and the second figure of the root, $.5 = r$. We then find $\delta'' = 5.5$, $\delta'' r = 2.75$, whence we obtain the complete divisor 10.75, the subtrahend 5.375, and a new dividend, .625000. Next, adding 2.75, the last correction .25, the square of the last figure of the root, and 10.75, the last complete divisor, we have 13.7500, the new trial divisor. Having next obtained 4, the third figure of the root, we annex it to twice the former figure, making 1.04, which we add to the 5.5, and thus obtain 6.54, the first factor of the new correction; and so on, as in the following operation: —

δ''	$\delta'' r$	δ'	3.544 +, root. 12 6
5.5	2.75	8.00	6.000
1.04	.25	10.75	5.375
6.54	.2616	13.7500	.625000
84	16	14.0116	.560464
6.624	.026496	14.274800	64536000
		14.301296	57205184
			7330816

In practice, it will be found that the location of the corresponding terms in columns $\delta'' r$ and δ', admits of the requisite facility in addition.

To continue the operation by decimal contraction, we observe that in any application of decimal contraction every multiplicand should have at least one redundant figure; that is, one place more than is necessary to obtain the required number of places in the product. This figure may be multiplied mentally, the tens of the product being carried to the product of the next figure, according to the common rule for contracted decimal multiplication. The suggestions which follow will enable us to secure this condition.

1st. For each new figure in the root, the terms in the operation extend to the right 1 place in the column marked δ'', 2 places in the column marked δ', and 3 places in the column of dividends. Hence,

2d. If at any point we omit to extend the dividend to a new period, we must *shorten* the corresponding term in column δ' 1 place, and the corresponding term in column δ'' 2 places.

3d. If, however, for the first contraction in columns δ' and δ'', we simply omit the *extended parts*, and afterward contract according to the precept just given, each contracted multiplicand will have *one redundant figure.*

It will be observed, that, after the contraction commences, the square of the last figure in the root is not used in forming the new trial divisors, for it is of an order inferior to the lowest in the divisor to be found.

In accordance with these suggestions, the entire operation will appear as follows, decimal points being omitted.

			3.5445115283 ±, root.
			12
δ''	$\delta'' r$	δ'	6
55	275	800	6000
104	25	1075	5375
654	2216	137500	625000
84	16	140116	560464
6624	26496	14274800	64536000
85	16	14301296	57205184
6632	3316	14327808	7330816
		14331124	7165562
66	7	1433444	165254
		1433451	143345
		143346	21909
			14335
		14335	7574
			7168
		1434	406
			287
		143	119
			114
		14	5
			4

Let it now be required to find one root of the equation,

$$y^3 + .34785634\,y^2 - 1.03562855\,y = -65.85732648.$$

In examples like this it will be convenient to write the terms in column $\delta'' r$ with units of like order in the same column.

Having found $m = -4$, and computed $\delta = -54.29178436$, and $\delta' = 44.18152073$, the operation will appear as follows: —

			—4.245574430 ±, root.
			—65.85732648
δ''	$\delta'' r$	δ'	—54.29178436
—11.85214366	+2.37042873	+44.18152073	—11.56554212
.44	4	46.55194946	9.31038989
12.292144	.4916858	48.9223782	2.25515223
85	16	49.4140640	1.97656256
12.3771	.061886	49.907350	.27858967
105	25	49.969236	.24984618
12.39	.00620	50.03115	2874349
		50.03735	2501868
12.	.0008	50.0436	372481
		50.0444	350311
		50.045	22170
			20018
		50.05	2152
			2002
		50.1	150
			150
		50.	0

The root thus obtained will always be correct to as many places as are found in any term in the column of dividends.

Geometrical Method.

The formulas for these operations may be deduced geometrically, by a process very similar to that required for the corresponding rule for cube root. For this purpose, let the following problem be proposed: —

Required the dimensions of a rectangular parallelopipedon whose breadth and length exceed its height by d and d' respectively, and whose solidity is g.

Let $y =$ the height; then $y + d =$ the breadth, and $y + d' =$ the length. Multiplying together the three dimensions, we have

$$(1) \qquad y^3 + (d + d')y^2 + dd'y = q.$$

Put $s = d + d'$, and $p = dd'$; then

$$(2) \qquad y^3 + sy^2 + py = q,$$

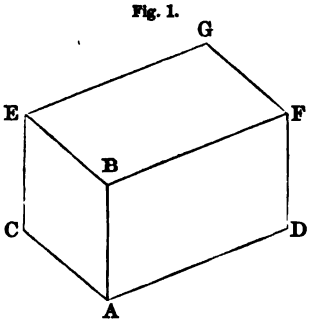
the general form employed in the foregoing pages.

We will suppose the required root of this equation to be irrational, and let $m =$ the initial figure. Since the entire root is to be the height of a right prism whose several dimensions are as given in the problem, we will form a right prism (Fig. 1), having $AB = m$, the height; $AC = m + d$, the breadth; and $AD = m + d'$, the length. The solidity of the prism is

$$m(m+d)(m+d') = m^3 + (d+d')m^2 + dd'm = m^3 + sm^2 + pm = \delta.$$

Subtracting the solid contents of the prism from that of the required prism, we have, for a remainder, $q - \delta$. If we now enlarge our prism (Fig. 1) by the addition of this remainder, its solidity will be that of the required prism. To preserve the given relations, the addition must be made upon three adjacent sides or faces, in the form of a covering of uniform thickness. The addition will therefore be composed of three flat prisms to cover the three faces (Fig. 2), and three oblong prisms and one cube to fill the vacancies at the edges and corner (Fig. 3). Now the thickness of this enlargement will be additional height of the prism, and consequently the remaining part of the root. Hence, to obtain a second figure of the root, we may divide the remainder, $q - \delta$, by the sum of the areas of the three flat prisms (Fig. 2), as a *trial divi-*

Fig. 1.



sor. These areas are the same as the three faces of Fig. 1, and are as follows: —

$$ABEC = m(m+d) \qquad = m^2 + dm.$$
$$ABFD = m(m+d') \qquad = m^2 + d'm,$$
$$BFGE = (m+d)(m+d') = m^2 + (d+d')m + dd'.$$

Adding these areas, we find the sum to be

$$3m^2 + 2sm + p = \delta'.$$

Represent by r the figure obtained by this trial divisor, which will express the common width of the four remaining prisms (Fig. 3).

Fig. 2.

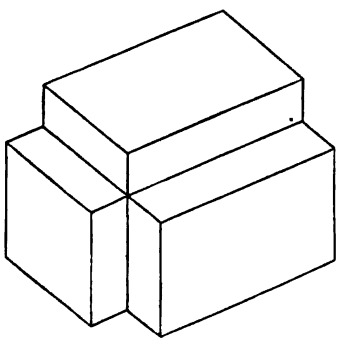
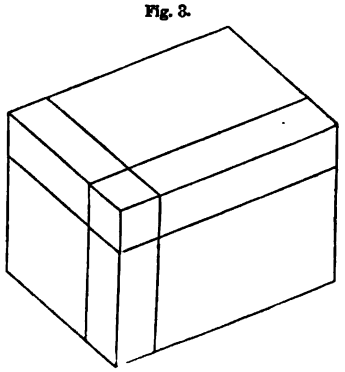


Fig. 3.



To complete the divisor, we must add to the trial divisor the sum of the areas of these four remaining prisms. The aggregate length of these prisms is $m + (m+d) + (m+d') + r = 3m + s + r = \delta''$; their area will therefore be $\delta'' r$, which is the correction required.

Having completed the divisor, and formed a new remainder, we must proceed to make a new enlargement of the prism. The trial divisor to obtain the thickness of this second enlargement, or the next figure of the root, will be the area of three new flat prisms to cover the three faces of the prism already formed; and this surface (Fig. 3), is composed of one face of each of the former flat prisms, two faces of each of the oblong prisms, and three faces of

the cube at the corner. But we have in the last complete divisor one face of each of the flat prisms, oblong prisms, and cube; and in the last correction, one face of each of the oblong prisms and cube; and in the square of the last figure of the root, a third face of the cube; and these are the three quantities which were added together to form the new trial divisor, as before explained.

We have thus derived the algebraical formulas before used, by a process entirely geometrical. To adapt this explanation of the rule to the case of a negative root, we have only to give the prism the proper location relative to three rectangular co-ordinates, according to the principles of analytical geometry.

MOTION ON AN INCLINED PLANE.

(Prepared for Elementary Instruction.)

By Thomas Sherwin, Principal, English High School, Boston.

We have seen, in the Theory of the Inclined Plane, No. V. Vol. II., that, when the power acts parallel to the plane, θ being the angle of the elevation, h, the height, and l, the length of the plane, R being 1, $P : W = \sin\theta : 1 = h : l$.

Now P, in this case, is equal to that part of the weight which urges the body down the plane. If, then, the body were unrestrained, it would, the elevation of the plane remaining the same, be subject to a constant force which would produce a uniformly accelerated velocity; and it is evident that this accelerating force bears to that of a body falling freely the same ratio that the equilibrating power bears to the weight. Therefore, calling g' the accelerating force down the plane, we have $g' : g = \sin\theta : 1 = h : l$; $\therefore$

$$(1) \qquad g' = g\sin\theta = \frac{hg}{l}.$$

Since the force which urges a body down a plane is the same as that which causes a body to fall freely, except that it is of less intensity, it follows that the motion of a body down such a plane is subject to the same laws as that of falling bodies. Hence, on the same plane,

(2) $$t : t' = v : v',$$

(3) $$s : s' = t^2 : t'^2 = v^2 : v'^2,$$

(4) $$s_1 : s_2 : s_3, \text{ \&c.} = 1 : 3 : 5, \text{ \&c.};$$

that is,

The times are as the final velocities; the spaces are as the squares of the times, or as the squares of the final velocities; and the spaces described in equal successive portions of time are as the successive odd numbers, 1, 3, 5, 7, &c.

Since $g' = g \sin\theta = \frac{hg}{l}$, if we substitute these values of g' instead of g in the formulæ for falling bodies, we shall have corresponding formulæ for the motion of bodies on inclined planes. Hence,

(5) $v = gt \sin\theta = \frac{hgt}{l}$; (6) $t = \frac{v}{g \sin\theta} = \frac{lv}{hg}$

(7) $s = \frac{1}{2} g t^2 \sin\theta = \frac{1}{2} \frac{hgt^2}{l}$; (8) $t = \sqrt{\frac{2s}{g \sin\theta}} = \sqrt{\frac{2ls}{hg}}$;

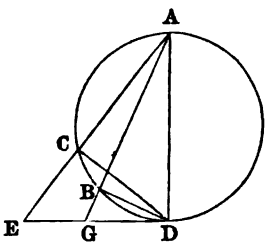
(9) $s = \frac{v^2}{2g \sin\theta} = \frac{lv^2}{2hg}$; (10) $v = \sqrt{2gs \sin\theta} = \sqrt{\frac{2hgs}{l}}$.

Formulæ may be easily obtained for bodies *projected* down or up inclined planes. By reference to Articles IV. and V. of Falling Bodies, the student will find no difficulty in obtaining these formulæ.

II. Let u be the velocity which a body acquires in falling freely through the height h, and v the velocity which the body acquires in moving down l, the length of a plane. Then $u = \sqrt{2gh}$, and, from (10), since s equals l, $v = \sqrt{\frac{2hgl}{l}} = \sqrt{2gh}$; $\therefore u = v$. Hence,

A body acquires the same velocity in moving down the length of an inclined plane, as it would acquire in falling freely through the height of the plane.

Let t and t' be the times in which a body would descend the inclined planes AG and AE as far as to the perpendiculars BD and CD drawn from D to these planes, and let the elevation of these planes be θ and θ'. Then, since AB and AC are the spaces, we have



$$AB = \tfrac{1}{2} g t^2 \sin\theta, \quad \text{and} \quad AC = \tfrac{1}{2} g t'^2 \sin\theta'.$$

But, in the right-angled triangles ABD and ACD,

$$AB = AD \cos BAD = AD \sin\theta,$$

and
$$AC = AD \cos CAD = AD \sin\theta';$$

$$\therefore AD \sin\theta = \tfrac{1}{2} g t^2 \sin\theta, \qquad \therefore (1)\ AD = \tfrac{1}{2} g t^2;$$

and
$$AD \sin\theta' = \tfrac{1}{2} g t'^2 \sin\theta', \qquad \therefore (2)\ AD = \tfrac{1}{2} g t'^2;$$

hence, from (1) and (2),

$$\tfrac{1}{2} g t^2 = \tfrac{1}{2} g t'^2, \qquad \therefore t = t',$$

Hence, the times of descent through AB and AC are the same.

If upon AD as a diameter the circumference of a circle be described, it will pass through the vertices of the right angles B and C, and AB and AC are chords drawn from A. Hence:

If any number of chords are drawn from either extremity of the vertical diameter of a circle, a heavy body would move down these chords or fall freely through the vertical diameter in the same time.

III. If we call l and l' the lengths of two inclined planes, h and h' their heights, and t and t' the times of descent through these lengths since l and l' are the spaces, we have from formula (8), and from the analogous formula $t' = \sqrt{\frac{2 l' s}{h' g}}$, by putting l and l' for s,

$$t = \frac{l\sqrt{2}}{\sqrt{h g}}, \qquad \text{and} \qquad t' = \frac{l'\sqrt{2}}{\sqrt{h' g}}; \qquad \therefore t : t' = \frac{l\sqrt{2}}{\sqrt{g h}} : \frac{l'\sqrt{2}}{\sqrt{g h'}}.$$

Dividing both terms of the last ratio by $\sqrt{2}$, and multiplying by $\sqrt{g}$, we have

(1) $$t : t' = \frac{l}{\sqrt{h}} : \frac{l'}{\sqrt{h'}}.$$

Hence,

In two different inclined planes, the times of descent through the lengths of the planes are as the lengths divided by the square roots of the heights.

If $h = h'$, proportion (1) becomes

(2) $$t : t' = l : l'.$$

If in the same proportion $l = l'$, it becomes $t : t' = \frac{1}{\sqrt{h}} : \frac{1}{\sqrt{h'}}$;

or, multiplying both terms of the last ratio by $\sqrt{h}$ and $\sqrt{h'}$,

(3) $$t : t' = \sqrt{h'} : \sqrt{h}.$$

Hence, from (2) and (3),

When in two inclined planes the heights are the same, the times of descent are as the lengths; and when the lengths are the same, the times of descent are inversely as the square root of the heights.

It may be proved in a similar way that the times of descent down the chords CD and BD are equal, and as the diameter itself may be regarded as a chord, the times of descent would each be equal to that in which a body would fall freely through this diameter.

THE THEORY OF THE GYROSCOPE.

By Theodore G. Ellis, Civil Engineer, Boston.

The interest excited in this little instrument, by its wonderful power of seeming to overcome the force of gravity, has induced the following attempt at an explanation of its principles in as concise and simple a manner as possible, with a hope that it may call

out the investigations of others who have made experiments or researches upon this interesting subject.

To have an apparatus in every way suited to our investigations, with a view to their being more readily understood, let us suppose the ordinary Gyroscope, consisting of a wheel, axle, and supporting ring, to be hung in another ring by pivots at right angles to the direction of the axis of the wheel; and this second ring to be supported also by pivots, at right angles to the former ones, upon a stand capable of turning upon a vertical axis, which, if produced, would pass through the centre of the wheel. The lines joining the opposite points in the rings would also pass through the centre of the wheel.

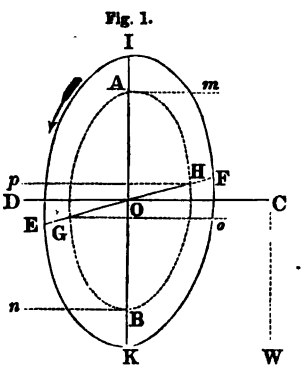
We then have the Gyroscope hung in gimbals precisely like a mariner's compass, except the additional contrivance for turning upon a vertical axis.

This mode of hanging the wheel admits of its motion freely in all possible directions around its centre, except when the horizontal axis of the wheel, the vertical axis, and all the points of suspension lie in the same vertical plane. In this position, motion of the wheel round its centre in that plane cannot take place. In our present examination we will suppose this position not to occur, as in all others motion will be free in all directions around the centre of the wheel.

For the sake of simplicity, we will also suppose the revolving wheel to be a circular disk, suspended on an axis of no appreciable weight compared with the weight of the wheel.

Let $IEKF$ (Fig. 1) be a circular disk, suspended as above described, and revolving with a given velocity in the direction of the arrow. Let DC and EF be horizontal, and IK vertical. At the end, C, of the axis, DC, let there be a force, W, acting downwards parallel to IK.

Fig. 1.



Then will each particle of matter in the wheel above the horizontal diameter, EF, have a tendency to move in the direction Am parallel to DC, and each particle of matter below EF will have a tendency to move in the opposite direction, Bn.

We see, therefore, that at any one instant of time, all the particles of matter above CF will be acted upon by a force in the direction Am, but when the wheel has performed a half revolution, these same particles are acted upon by a force in the direction Bn, exactly opposing and neutralizing the former. And so with the half circle below EF, it is at one instant acted upon in the direction Bn, and the next, when it has made half a revolution, it is acted upon in the direction Am. Thus each particle of which the wheel is composed receives in rapid succession impulses in opposite directions, which, owing to its inertia and the rigidity of the wheel, it constantly resists.

Let us take a particle in the vertical diameter, as at A, and follow it round the circle. During its passage through the quadrant AG it is continually acted upon by a succession of forces tending to move it in the direction Am, parallel to DC. After passing the horizontal diameter EF, from G to B it is acted upon by a succession of forces exactly equal to those above the horizontal diameter, but acting in an opposite direction, Bn, so that any motion or momentum acquired in passing through AG will be exactly counterbalanced and wholly destroyed in passing through GB, so far as relates to the directions Am and Bn; so that if the particle was at rest at A, relative to those directions, it will be again at rest on arriving at B.

The same effects as above described also evidently take place while the particle passes through BHA.

What is true for any one particle must be true for every other, and consequently for the whole wheel, and the forces above and below EF just balancing each other, no motion around EF results from the force, W, while the disk is revolving with a uniform velocity.

Having thus shown that the force, W, will not produce motion about the horizontal diameter, EF, which is at right angles to DC, the axis, DC, will remain in a horizontal plane. Motion, therefore, of the plane, $IEKF$, can only occur about the vertical axis IK, as we have the point O fixed by construction, being the centre of all motion of which the wheel is capable.

Let us now examine the effect of the forces acting on any particle of matter passing around the circle, with regard to the axis IK.

From A to G we have seen that it is acted upon by a force tending to move it in the direction Am; and from G to B by a force tending to move it in the direction Bn; and that these forces are equal and act in opposite directions. Now if a body is acted upon by any force for a definite period of time, motion will be produced in the direction of the force, and a certain velocity will be acquired; if an equal force then act on the same body in an opposite direction, for the same period of time, the velocity acquired is gradually retarded and the body left at rest.

This is precisely the case with a particle of matter passing through AGB or through BHA. A force tends to accelerate its motion in the direction Am from A to G, and from G to B it is gradually retarded by a force acting in the direction Bn. So also through BHA a force tends to accelerate its motion in the direction Bn, while passing through BH, and through HA it is again retarded and brought to rest by the force acting in the direction Am.

As the forces producing these motions around IK are equal on opposite sides of IK, and their centres of moment are equally

distant, they will produce motion round IK as an axis independently of the point, O, being fixed by construction.

Let us for a moment consider what would be the effect if this circular motion around IK could not take place. Suppose, for instance, that the axis CD were confined to a vertical plane, and to be in a horizontal position. In this case no horizontal circular motion could occur, so that all particles of matter of which the wheel is composed passing through EO downwards and through OF upwards would be brought to relative rest with regard to a horizontal plane. But immediately on passing EO and OF they would be acted upon by the opposite forces in the directions Am and Bn; motion would immediately take place around EF, and the weight W would fall.

We thus see that this motion around IK not only takes place, but is absolutely necessary to the equilibrium maintained at A and B.

We will now proceed to ascertain the time of rotation around IK. Suppose the circular disk to be divided into an indefinite number of small sectors, then will the centre of gravity of each be distant from the centre of the disk by two thirds of the radius of the disk. For convenience of demonstration we will consider all the weight of the disk to be in the circle joining these centres of gravity, calling the radius of this circle "radius of gravity," and limiting ourselves to such reasoning as will be consistent with the above supposition.

This method of considering the weight of the wheel to be in its circle of gravity has also the advantage of being applicable to wheels of all forms.

As the forces acting in the quadrants GB and HA are solely employed in bringing to a state of rest the motion acquired by each particle of matter in the quadrants AG and BH, they exercise no

influence upon the rotation of the wheel around IK, which rotation is therefore solely due to the motion acquired in the quadrants AG and BH.

Again, each particle of matter of which the wheel is composed is limited to the plane of revolution of the wheel, so that in passing through AG or BH it will not admit of being uniformly accelerated, but will move in successive instants of time through spaces proportional to the sine of the angular distance from IK. But as in passing through AG or BH each quantity of motion lost by one particle on account of its being confined to the plane of the wheel, is distributed among all others lying on the same side of IK, the total amount of motion for any one particle is the same as though it were uniformly accelerated while passing through AG or BH.

Referring again to Fig. 1, let

r = AO = radius of gravity.
a = OC.
t = time employed in revolving.
g = force of gravity = 32.2.
π = ratio of circumference to diameter.
W = force, in units of weight, at C.
W' = weight of wheel.

To find the accelerating force, F, acting on each particle of the wheel and turning it round IK, we proceed as follows:—

We have W = number of units of weight acting at a radius of a tending to turn the wheel on GH. We have also the distance from O to the centres of moment of the semicircles

$$GAH \text{ and } GBH = \frac{2r}{\pi}.$$

Denoting by F the force in units of weight acting on the centres of moment of the semicircles GAH, GBH, we have

$$F \times \frac{2r}{\pi} = aW, \qquad F = \frac{\pi a W}{2r}.$$

If then the weights W and W' were equal, we should have the accelerating force, F, acting on the wheel and the force of gravity, g, proportional to the pressures F and W, or

$$F : g = \frac{\pi a W}{2r} : W, \qquad F = \frac{\pi g a}{2r}.$$

But if the weights W and W' are different, the force F will vary inversely as the proportion of W' to W, or directly as $\frac{W}{W'}$; hence,

$$F = \frac{\pi g a}{2r} \times \frac{W}{W'}.$$

By the law of bodies moving under the influence of a continuous force, the space described equals $\frac{1}{2} t^2 F$. We have, therefore,

$$(1) \qquad s = \tfrac{1}{2} t^2 \times \frac{\pi g a W}{2 r W'} = \frac{\pi g t^2 a W}{4 r W'}.$$

Representing by T the time required for one whole rotation about IK, we have,

$$T : t = 2\pi r : \frac{\pi g t^2 a W}{4 r W'},$$

$$(2) \qquad T = \frac{8 r^2 W'}{g t a W}.$$

If $n =$ number of revolutions of wheel per second, then $4t$, equal to the time of one revolution, $= \frac{1}{n}$, and $t = \frac{1}{4n}$. Substituting in above formula, we have

$$(3) \qquad T = \frac{32 n r^2 W'}{g a W}.$$

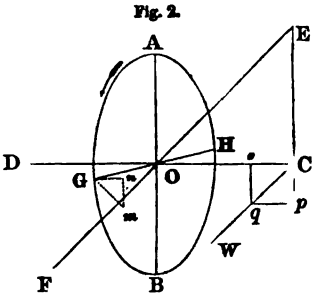
In these formulæ the units of length and time are feet and seconds. Of weight, any unit may be taken.

Let us now consider the case where the axis CD is not horizontal, but inclined at any angle to the horizon except vertical, when the force W would pass through the centre O.

We will suppose the Gyroscope suspended as before, and revolving with a uniform velocity.

Let AB, CD, and GH (Fig. 2) represent the same lines as before, AB and CD lying in the same vertical plane, and GH being horizontal. Let W be the weight hanging vertically from the end of the inclined axis CD. Draw the vertical line EF through the centre of the wheel, O. Represent the weight W by Cq, and resolve this force into two others lying in the vertical plane which passes through EF and CW; one force being at right angles to, and the other coinciding with the axis CD. Completing the parallelogram we have one force $= Cp$, and the other $= Co$. Produce Cp to E.

Fig. 2.



The force Co acting through the centre of the wheel has no tendency to turn it in any direction. The force Cp, acting at right angles to CD, and tending to turn the wheel on the horizontal line GH, at right angles to CD, would, as we have before shown, cause the wheel to rotate on AB if it continued to act in a direction parallel to Cp. That is, the point G would move in the direction Gn. But this force is constantly changing its direction, as it always lies in a vertical plane passing through EF; and as it constantly tends to turn the wheel in the vertical plane in which it lies, the line GH, being always at right angles to this plane, must remain horizontal, and the points G and H will move in a horizontal circle around O. The angle EOC will remain constant, and the wheel will rotate about the vertical line EF parallel to CW.

Let us now suppose the point C to revolve around EF the indefinitely small distance Cx. In passing through this indefinitely small space we may consider two forces to have been acting at the point

C, one of which is continually parallel to Cp, causing the wheel to rotate on AB, and the other operating to turn AB and GH on CD as an axis through the small angle AEx.

From G lay off Gn parallel to DC equal to the distance the point G would move under the influence of the parallel force Cp. Draw nm perpendicular to Gn, and make Gm horizontal. Then will Gn and nm represent the distances the point G would be moved by the two forces acting at the point C. Gm is the resultant, and represents the distance actually passed through by the point G.

In the triangles Gnm and Cqp we have the sides of one respectively perpendicular to the sides of the other in direction; therefore they are similar, and we have

$$Gn : Gm = Cp : Cq.$$

We see, from equation (1) previously obtained, that the space described in rotating by the influence of a force acting at right angles to CD is proportional to the weight W. If, therefore, Gn is the distance revolved under the influence of the force Cp, Gm would be the distance revolved under the influence of the force $Cq = W$ acting at right angles to CD, and is therefore the same distance that would be described by the point G when CD is horizontal.

As the proof of the above would be similar for any inclination below the horizontal line, we see that for all inclinations of CD we have the angular velocity and time of rotation about the vertical axis the same.

From the foregoing we may deduce the general law of the Gyroscope, which will account for all the phenomena exhibited by different experiments, viz.: —

A revolving body, acted upon by a force passing through its axis of revolution, will rotate with a new motion round an axis passing through its centre of motion parallel to the direction of the force.

The time of rotation will be directly proportional to the velocity of revolution, the square of the radius of gravity in the plane of revolution, and the weight of the body; and inversely proportional to the force, and the distance on the axis from the centre of the body to the line in which the force acts.

The case in which resistance occurs to the rotating motion comes under the above general law by considering the reaction from the resistance to be a force.

Let us see how this law operates in the ordinary form of toy gyroscope, apparently supporting its own weight and rotating around a fixed support.

Here we have the whole weight of the wheel and ring acting downwards by the force of gravity; but as the wheel does not yield to its influence, on account of its being obliged to turn out of its revolving plane to fall, which motion we have seen is resisted by a revolving body, the whole weight is transferred to the point of support, which reacts in an upward direction with an equal force to sustain it. Here we have a force acting on the end of the axle, equal to the weight of the wheel and ring, tending to turn the revolving wheel on its horizontal diameter. This we have seen will cause it to rotate on a vertical axis. But as the point of suspension is fixed, the rotation of the wheel on a vertical axis reacts on the point of support, and the wheel itself moves instead of the point at which the upward force acts. And as one rotation of the wheel on its vertical axis will cause one complete turn around the point of support, the time employed in making the latter will be given by the formula (3) previously found, viz.: —

$$T = \frac{32\, n\, r^2\, W'}{g\, a\, W}.$$

In this case W is the weight of the wheel and ring, and W' the weight of the wheel alone.

When the Gyroscope is suspended by a cord instead of the fixed support, it also appears to turn about the point of suspension. The reason of this is, that the whole weight of the apparatus being transferred as before to the point of suspension, that point is in stable equilibrium only when the cord hangs vertically, and the resistance to motion, to any distance out of this vertical line, is very much greater than the resistance of the air to the passage of the revolving wheel, so that the slight reaction on the point of suspension necessary to cause rotation round this point will move it but a small distance, and cause the cord to vary but slightly from a vertical position.

Finally, let us examine into the cause of the gradual falling of the weight attached to the axis of the Gyroscope.

We have already seen that when the wheel revolves uniformly and without friction, there is no yielding in the direction of the force, it being entirely expended in producing the rotation round the vertical axis.

That the falling of the weight is due to friction, including under this term all causes of resistance, may be shown as follows.

Let us consider the friction as operating in two ways: First, by causing a retardation of the velocity of the revolving wheel, and, secondly, by the resistance it offers to motion at the point of support.

It is evident, from our foregoing reasoning, that if a particle of matter is longer in passing through the second quadrant on either side of AB than through the first, it will more than have its motion arrested, and will acquire a motion in an opposite direction, and will not be at rest on arriving at A or B; and by just so much distance as it has moved through will the wheel be turned on its horizontal diameter, thus allowing the weight to fall through a corresponding angle.

We have also seen that any force arresting the rotation about AB causes the weight to fall. The friction upon the point of support, though not actually stopping the rotation of the wheel, operates in the same manner. The particles of matter in the first quadrant are prevented from acquiring the full amount of motion due to the force, and thus in the second quadrant the motion is more than counterbalanced, so that instead of being at rest at the points A and B, motion is acquired in the opposite direction; and by just the distance moved through, after coming to rest before arriving at A or B, will the wheel be turned on its horizontal diameter and the weight will fall by a corresponding amount.

A COMPLETE CATALOGUE OF THE WRITINGS OF SIR JOHN HERSCHEL.

PREPARED FOR THE MATHEMATICAL MONTHLY BY THE AUTHOR.

1812. NICHOLSON'S Journal, XXXI. p. 133, No. 142. Analytical Formulæ by "A Lover of the Modern Analysis."
1812. Ibid., XXXII. p. 13. A letter signed "Analyticus." (Trigonometrical Formulæ.)
1812. LEYBOURNE'S Mathematical Repository for 1814, Part II p 58. New Properties of the Conic Sections.
1812. Phil. Trans. Royal Soc. London, 1813. On a Remarkable Application of COTES'S Theorem.
1813. On Trigonometrical Series, ... &c., ... Mem. Analyt. Soc. Cambridge.
1813. On Equations of Differences and their Application to the Determination of Functions under Given Conditions. Mem. Analyt. Soc. Cambridge. N. B. — The latter portion of this Part III. on Functional Equations is all wrong.
1814. Phil. Trans. Royal Soc. London, 1814. Consideration of Various Points of Analysis.
1814. Problems 362, 363, 364, 365, 366, 367, 368, 369, LEYBOURNE'S Mathematical Repository. — Also (I believe somewhere in that Repository), Properties of the Catenary, analogous to the Trigonometrical Formulæ in the Circle, and a number of other problems, &c., in other parts of that Repository.
1815. Phil. Trans. Royal Soc. London, 1816. On the Development of Exponential Functions together with several new Theorems relating to Finite Differences.
1816. The notes N, O, P, Q, in the Translation of LACROIX'S 8vo Treatise on the Differential and Integral Calculus, by Messrs. PEACOCK, HERSCHEL, and BABBAGE, with an original Appendix on Differences.
1817. The article "Isoperimetrical Problems" in the Edinburgh Cyclopædia.
1818. The article "Mathematics in the Edinburgh Cyclopædia, XIII., Parts 1, 2.
1818. On Circulating Functions, and on the Integration of a Class of Equations of Finite Differences into which they enter as Coefficients. Read before the Royal Soc., Feb. 19. Phil. Trans.
1819. The Preface to SPENCE'S Mathematical Essays. (Edited by Sir J. H.)
1819. Original Notes on SPENCE'S Mathematical Essays, published in 1819.
1819. On the Hyposulphurous Acid and its Compounds. BREWSTER'S Edinburgh Phil. Journal, Vol I., p. 8.
1819. A Notice in BREWSTER and JAMIESON'S Journal of Sir W. HERSCHEL'S Paper on the Distance of the Fixed Stars.

1819. On some Optical Phenomena exhibited by Mother of Pearl, dependent on its Internal Structure. BREWSTER and JAMIESON's Journal, I. p 396.
1819. On the Application of a new Mode of Analysis to the Theory and Summation of certain extensive Classes of Series. BREWSTER and JAMIESON's Edinburgh Phil. Journal, Vol. II p. 23.
1819. Some Additional Facts relating to the Habitudes of the Hyposulphurous Acid and its Union with Metallic Oxides. BREWSTER and JAMIESON's Phil. Journal, Vol. II. p. 154.
1819. Refractive Indices of Hyposulphite of Soda and Silver and of Nitrate of Lead. BREWSTER and JAMIESON's Phil. Journal, Vol. II. p. 184.
1820. A Collection of Examples of Applications of the Calculus of Finite Differences.
1820. On certain Remarkable Instances of Deviation from Newton's Scale in the Tints developed by Crystals with one Axis of Double Refraction on Exposure to Polarized Light. Cambridge University Phil. Trans., Vol. I.
1820. On the Rotation Impressed by Plates of Rock Crystal on the Planes of Polarization of the Rays of Light as connected with Certain Peculiarities in its Crystallization. Cambridge University Phil. Trans , Vol. I.
1820. On the Reduction of Certain Classes of Functional Equations to Equations of Finite Differences. Read March 6th. Trans. of the Cambridge Phil. Soc.
1821. On a Remarkable Peculiarity in the Law of Extraordinary Refraction of differently colored Rays exhibited by certain varieties of Apophyllite. Cambridge University Phil. Trans., Vol. I.
1821. On the Aberrations of Compound Lenses and Object Glasses. Phil. Trans. Royal Soc. London.
1821. Auszug Einer Briefe von Sir J. F. W. H. Astron. Nachr., No. 2, p. 30.
1821. On the Separation of Iron from other Metals. Read April 5th, Phil. Trans.
1821. A Letter to Dr. BREWSTER on the Practical Rules for the Determination of the Radii of a Double Achromatic Object Glass. Edinburgh Phil. Journal.
1822. On the Absorption of Light by Colored Media, and on the Colors of the Prismatic Spectrum exhibited by various Flames, with an Account of a ready Mode of obtaining the absolute Dispersive Power of any Medium by direct Experiment. Edinburgh Phil. Trans., Vol. IX. p. 446.
1823. Auszug Einer Briefe, &c. Astron. Nachr., No. 44.
1823. The Article "Physical Astronomy" in the Encyclopædia Metropolitana.
1823. Subsidiary Tables for facilitating the Computations of Annual Tables of the Apparent Places of 46 Principal Fixed Stars computed by order of the Council of the Astron. Soc., &c., &c. Mem. Astron. Soc., Vol. I., published 1824.
1824. Observations of the Apparent Distances and Positions of 380 Double and Triple Stars, by J. F. W. H. and J. SOUTH. Phil. Trans , 1825, Part III.
1824. A Notice of AMICI's Double-Image Micrometer. Notice read to Astron. Soc. Phil. Magazine.
1824. On a New Method of computing Occultations of the Fixed Stars. Communicated in a Letter to CHARLES BABBAGE. Read Jan. 10th, 1823. Mem. Astron. Soc.
1824. On Certain Motions produced in Fluid Conductors when transmitting the Electric Current. (Bakerian Lecture.) Read before the Royal Soc., Feb. 12th. Phil. Trans.
1825. On the Mechanical Effects produced when a Conducting Liquid is electrified in contact with Mercury. BREWSTER's Edinburgh Journal of Science, Vol. II. p. 193.
1825. Analysis of Dr. BREWSTER's Paper on the Monochromatic Lamp, and of the Absorption of the Prismatic Rays by Colored Media, &c. BREWSTER's Edinburgh Journal, Vol. II. p. 344.
1825. Extract of a Letter from J. F. W. HERSCHEL to Dr. RITCHIE. (This contains the first published account of my Principle of Actinometry.) Phil. Jour., Vol. III. p. 107.
1825. The notes signed (H) in Mr. SOUTH's Paper on Double Stars. Phil. Trans. Royal Soc. 1823.
1825. Abstract of a Memoir by M. PLANA. Proceedings of Astron. Soc. London, 1825, Phil. Magazine.
1825. An Account of the Repetition of M. ARAGO's Experiments on the Magnetism manifested by various Substances during the Act of Rotation. Read before the Royal Soc., June 16th. Phil. Trans.
1825. Extract of a Letter to M. SCHUMACHER, dated Aug. 15th, on Mr. FRAUNHOFER's Memoir on the Inferiority of Reflecting Telescopes when compared with Refractors.
1825. Approximate places of 321 new Double and Triple Stars, for January 1st, 1825, with their estimated Angles of Position, Distances, Magnitudes, and other Particulars. Astron. Soc. of London, Vol. II.
1826. An Account of a Series of Observations made in the Summer of the Year 1825, for the Purpose of Determining the Difference of the Meridians of the Royal Observatories of Greenwich and Paris. Read before the Royal Society, Jan. 12th. Phil. Trans.
1826. On the Parallax of the Fixed Stars. Read before the Royal Society, March 9 and 16. Phil. Trans.

1827. Correction of an Error in a Paper on the Parallax of the Fixed Stars. Phil. Trans. Royal Soc.

1827. A Paper read at a Meeting of the Board of Longitude. Printed by order of the Board. On Reform of the Nautical Almanac.

1827. The article "Light" in the Encyclopædia Metropolitana. This has been translated into French and German.

1827. An Address delivered at a Special General Meeting of the Astronomical Society of London, on April 11th, 1827, on presenting the Honorary Medals to F. BAILY, Esq., Lieut. W. S. STRATFORD, R. N., and Col. MARK BEAUFOY.

1827. An Account of Observations made with a Twenty-feet Reflecting Telescope, containing a Second Catalogue of 295 New Double and Triple Stars (reduced to the beginning of 1830); together with some observations of Double Stars previously known. Memoirs of the Astron. Soc. London., Vol. III.

1828. Third series of Observations with a Twenty-feet Reflector, containing a Catalogue of 384 New Double Stars (reduced to the beginning of 1828); together with some Observations of Double Stars previously known. Memoirs of the Astron. Soc. London, Vol. III.

1828. An Address on the Occasion of the Delivery of the Honorary Medals of the Astronomical Society of London, on Feb. 8th, 1828, to Lieut. Gen. Sir T. MACDOUGAL BRISBANE, K.C.B., and JAMES DUNLOP, Esq.

1829. An Address delivered at the Anniversary Meeting of the Astronomical Society of London, Feb. 13th, 1829, on Presenting the Honorary Medals to Rev. W. PEARSON, Professor BESSEL, and Professor SCHUMACHER.

1829. The Article "Sound" in the Encyclopædia Metropolitana, Vol. IV., for 1829–30.

1830. Extract of a Letter from Sir J. F. W. H. to M. QUETELET, on the "Roccellic Acid." Read at the Royal Academy of Sciences, Brussels, May 22d, 1830. Bulletins of the Academy.

1830. On the Study of Natural Philosophy, a Preliminary Discourse to LARDNER'S Cabinet Cyclopædia.

1830. Fourth Series of Observations with a Twenty-feet Reflector; containing the Places, Descriptions, and Measures of 1236 Double Stars (reduced to the beginning of 1830); the greater part of them not previously described. Memoirs of the Astron. Soc. London, Vol. IV.

1832. Fifth Catalogue of Double Stars observed at Slough in the years 1830 and 1831 with the Twenty-feet Reflector; containing the Places, Descriptions, and measured Angles of Positions of 2007 of those Objects, of which 1304 have not been found described in any previous collection; the whole reduced to the Epoch 1830. Communicated to the Royal Astron. Soc., June 7th, 1832. Royal Astron. Soc., Vol. VI.

1832. Micrometrical Measures of 364 Double Stars, with a Seven-feet Equatorial Achromatic Telescope; taken at Slough, in the years 1828, 1829, 1830. Memoirs of the Royal Astron. Soc., Vol V. Part I.

1832. On the Astronomical Causes which may influence Geological Phenomena. Read Dec. 15th, 1830. Trans. Geological Soc. London, 1832.

1832. Observations of BIELA'S Comet. Read Nov. 9, 1832. Royal Astron. Soc., Vol. VI.

1832. On the Action of Light in determining the Precipitation of Muriate of Platina by Lime-water, in a Letter to Dr. DAUBENY. Read before the British Association at Oxford, June 22d. London, Edinburgh, and Dublin Phil. Magazine, New Series, No. 1.

1832. Notice of BIELA'S Comet in a Letter to the "Times," Sept. 26th, 1832.

1832. On BIELA'S Comet, a 2d Letter to the "Times," Sept. 27th, 1832.

1832. Observations of BIELA'S Comet. Memoirs Astron. Soc. London, Vol VI. p. 99.

1832. Observations of δ Cygni, μ Aquarii, α Capricorni, and BIELA'S Comet. Astron. Nachr., No. 236.

1832. Description of a Machine for resolving by Inspection certain important Forms of Transcendental Equations. Read May 7th, 1832. Trans. of the Cambridge Phil. Soc.

1832. On the Investigation of the Orbits of Revolving Double Stars: being a Supplement to a Paper entitled "Micrometrical Measures of 364 Double Stars." Read Jan. 13th, 1832. Royal Astron. Soc., Vol. V.

1832. Lettre adressée à MM., les RÉDACTEURS des Annales de Chimie et de Physique, sur la Séparation de l'Oxide de Fer, et sur un nouveau Procédé pour effectuer la Purification complète de l'Oxide d'Urane; 24 Avril, 1832. Extrait des Annales de Chimie, Mars, 1832.

1833. An Address to the Subscribers to the Windsor and Eton Public Library and Reading-room. Published by KNIGHT, London.

1833. "Notice sur la Manière d'agir de l'Acide Nitrique sur la Fer." Annales de Chimie et de Physique, Tom. IV. p. 89. Lettre à MM., les RÉDACTEURS.

1833. Review of Mrs. SOMERVILLE'S Mechanism of the Heavens, and of NATHANIEL BOWDITCH'S Translation of the "Mécanique Céleste of Laplace." Quarterly Review, No. XCIV.

1833. A Treatise on Astronomy. Cabinet Encyclopædia, Part 43.

1833. Notices of the Elliptic Orbits of ξ Boötis and η Coronæ, with a second Approximation to the Orbit of γ Virginis. Read June 14th, 1833. Memoirs of the Royal Astron. Soc.. Vol. VI.

1833. A Letter addressed to R. PHILLIPS, Esq. &c., &c., dated January 12th, 1833, on a Remarkable

Deposition of Ice round the Decaying Stems of Vegetables during Frost. London and Edinburgh Phil. Magazine and Journal.

1833. On the Absorption of Light by Colored Media, viewed in connection with the Undulatory Theory. From the London and Edinburgh Phil. Magazine and Journal of Science.

1833. Observations of Nebulæ and Clusters of Stars, made at Slough, with a Twenty-feet Reflector, between the Years 1825 and 1833. Read Nov. 21, 1833. Phil. Trans.

1834. On the Satellites of Uranus. Read March 14th, 1834. Royal Astron. Soc., Vol. VIII.

1834. A List of Test Objects, principally Double Stars, arranged in Classes, for the Trial of Telescopes in various respects, as to Light, Distinctness, &c. Memoirs of the Royal Astron. Soc., Vol. VIII.

1834. A Second Series of Micrometrical Measures of Double Stars, chiefly performed with the Seven-feet Equatorial at Slough, in the Years 1831, 1832, and 1833. Memoirs of the Royal Astron. Soc., Vol. VIII.

1834. Letter to F. BAILY, Esq. Notices of Astron. Soc., Vol. III. p. 75.

1834. Meteorological Observations made at Feldhausen, C. G. H., Dec. 21st, 1834. Athenæum.

1835. Annual Report of the South African and Scientific Institution. Read July 11th, 1835.

1835. Instructions for Making and Registering Meteorological Observations in South Africa and other Countries in the South Seas. (Published in London by M. ROBERTSON in 1835.)

1835. Account of Proceedings at the Cape of Good Hope. SCHUMACHER's Astron. Nachr. No. 281.

1835. Meteorological Observations made at Feldhausen (Athenæum), March 21st, 1835.

1835. " " " " June 21st, 1835.

1835. " " " " Dec. 21st, 1835,

1836. Extract of a Letter to C. LYELL, Esq., dated Feb. 20th, 1836, "On the primary Cause of Geological Upheavals and Disturbances." Printed in C. BABBAGE's Ninth Bridgewater Treatise, p. 202.

1836. A Letter to F. BAILY, Esq., giving an Account of the present Appearance of γ Virginis. Notices of Astron. Soc, Vol. III. p. 197.

1836. Meteorological Observations made at Feldhausen, March 21st, 1836 (Athenæum).

1836. " " " " June 21st, 1836 "

1836. " " " " Sept. 21st, 1836 "

1836. " " " " Dec. 21st, 1836 "

1836. Report of the Meteorological Committee of the South African Literary and Philosophical Institution (published by the Institution).

1836. A Letter to Dr. ADAMSON of the South African College.

1836. Extract of a Letter to R. I. MURCHISON, Esq., on the *Primum Mobile* of Geological Disturbances. Printed by C. BABBAGE in his "Ninth Bridgewater Treatise," p 214.

1836. On the Cape of Good Hope Government Schools. Printed in Mr. C. BAMBURY's book on South Africa.

1836. Sixth Catalogue of Double Stars, observed at Slough, in the Years 1831 and 1832, with the Twenty-feet Reflector: containing the Places, Descriptions, and Measured Angles of Position of 286 of those Objects of which 105 have not been previously described. Reduced to the Epoch 1830. Memoirs Royal Astron. Soc., Vol. IX.

1837. A Letter to F. BAILY, Esq., dated Oct. 18th, 1836, on the Observations of the Comet of HALLEY, after the Perihelion Passage in 1836: made at Feldhausen, Cape of Good Hope. Read Jan. 13th, 1837. Trans. Royal Astron Soc.

1837. Meteorological Observations at Feldhausen in March 21st, 1837 (Athenæum).

1837. " " " June 21st, 1837 "

1837. " " " Sept. 21st, 1837 "

1837. " " " Dec. 21st, 1837 "

1837. Report of the Meteorological Committee. Read July 17th, 1837.

1838. Astron. Nachr., No. 354. Extract of a letter to Dr. MÄDLER on the sudden Change of Magnitude of the star η Argus

1838. Report of British Association, 1838. "Notices of Observations of Stars and Nebulæ observed at the Cape of Good Hope."

1838. Notice on HALLEY's Comet. Report of British Association, 1838.

1838. On a Method of Obtaining a great Increase of Light from a common Argand Burner. London, Edinburgh, and Dublin Phil. Magazine.

1839. Astron. Nachr., No. 372. Letter to Professor SCHUMACHER on the Variable Stars η Argus, α Hydræ, and α Cassiopeiæ; and on the comparative brightness of the Stars.

1839. Note on the Art of Photography; or on the Application of the Chemical Rays of Light to the purposes of Pictorial Representation. Notices of the Proceedings of the Royal Society of London. No. 37, p. 131.

1839. On the Variability of the Star α Cassiopeiæ. Notices of Royal Astron. Soc. May 10th, 1839.

1839. A Letter to the President of the British Association, giving an Account of the Production of analogous Colors by the Action of the colored prismatic Rays on Argentine Papers; and on a peculiar Property of the extreme red Rays. (Report of Proceedings of the British Association, 1839. Also Athenæum, No. 641.)

1839. Astron. Nachr., No. 405. "Requium of the Old Telescope," with a Translation into German by Madame MADLER.
1840. Address to the Astron. Soc. on occasion of delivery of a Medal to M. PLANA. (Notices of Astron. Soc., Feb. 1840.)
1840. On the Chemical Action of the Rays of the Solar Spectrum on Preparations of Silver and other Substances, metallic and non-metallic. Phil. Trans. Royal Soc. London, 1840.
1840. Report on the Subject of Terrestrial Magnetism in a Series of Resolutions, adopted by the British Association, at their Meeting in August, 1838, at Newcastle. Report of the British Association for the Advancement of Science for 1839.
1840. On the Variability and Periodical Nature of the Star α Orionis. Read Jan. 10th, 1840. Trans. Royal Astron. Soc.
1840. Review of GAUSS and W. WEBER's allgemeine theorie des Erdmagnet; and of GAUSS's Intensitas vis Magnetiæ. Quarterly Review, No. 131. June, 1840.
1840. Report of the Committee of Physics, including Meteorology. Royal Soc. London.
1841. A Letter to Professor STEVELLY, Sec. British Association, "On Processes for producing Colored Photographs." British Association, 1841.
1841. On Sir WM. HAMILTON's Observations of the Meteors of Aug. 9th, during the great Eruption of Vesuvius described by him. Athenæum, No. 721. Aug. 21st, 1841.
1841. An Address delivered on the Presentation of the Gold Medal of the Royal Astronomical Society to Professor BESSEL, at the Anniversary Meeting, Feb. 12th, 1841, for his Observations and Researches on the Parallax of 61 Cygni.
1841. On the Advantages to be attained by a Revision and Re-arrangement of the Constellations, with especial Reference to those of the Southern Hemisphere; and on the Principles upon which such Re-arrangement ought to be conducted. Memoirs of the Royal Astron. Soc. Vol. XII.
1841. Report of the Magnetic Committee of the British Association.
1841. Astron. Nachr., No. 442, On the Moon's Influence on the Weather.
1841. Remarks on certain Suggestions offered to the Commission for the Restoration of the lost Standard of Weights and Measures. Report of the Commission, 1841.
1841. Remarks on the Nomenclature of Coins, Weights, and Measures, and on some other Points connected with the Subject of the Restoration of the lost Standards. Report of the Commission.
1841. A Review of WHEWELL's History and Philosophy of the Inductive Sciences. Quarterly Review, No. CXXXV.
1842. On the Action of the Rays of the Solar Spectrum on Vegetable Colors; and on some new Photographic Processes. Read June 16th, 1842. Phil. Trans., Part II., for 1842.
1842. The Walk. Translated in the original Metre from the German of F. SCHILLER. For private circulation.
1842. Extracts from a letter addressed by Sir J. F. W HERSCHEL to Mr. MURCHISON, explanatory of the Phenomena of the Freezing Caves of Ilitzkaia Zatcheta. Notices of the Geological Soc. of London. Vol. ? p. 697.
1842. On some Phenomena observed on Glaciers, and on the Internal Temperature of large Masses of Ice or Snow, with some Remarks on the natural Ice Caves which occur below the Limit of Perpetual Snow. Notices of the Geological Soc. Vol. p. 699.
1843. An Account of the great Explosion for the Removal of the Round Down Cliff at Dover, in a Letter to the Editor of the Athenæum. Ath., Feb. 1843.
1843. First Notice of the Great Comet of 1843. Times Newspaper, March 20th or 21st, 1843.
1843. A second Letter to the Editor of the "Times" on do. do., March, 1843.
1843. A Letter to the Editor of the Morning Herald on do. do., March 25th or 26th, 1843
1843. Letter to the Editor of the "Times" on the Comet and the Zodiacal Light, April, 1843.
1843. Account of an Aurora Borealis. Athenæum, No. 811, p. 465.
1843. Notice of a remarkable Photographic Process by which Dormant Pictures are rendered susceptible of Development by the Breath, or by keeping in a moist Atmosphere. Proceedings of the British Association for August, 1843.
1843. On the further Increase of η Argus. Notices of the Royal Astron. Soc., Nov. 10th, 1843.
1843. Report of the Committee consisting of Sir JOHN HERSCHEL, the MASTER OF TRINITY COLLEGE, Cambridge, the DEAN OF ELY, Dr. LLOYD, and Colonel SABINE, appointed to conduct the Co-operation of the British Association in the System of Simultaneous Magnetical and Meteorological Observations. Report of the British Association for the Advancement of Science for 1843.
1843. A letter to the Editor of the Philosophical Magazine and Journal, dated Jan. 11th, 1843, On the Action of the Rays of the Solar Spectrum on the Daguerreotype Plate. From the London, Edinburgh, and Dublin Phil. Magazine and Journal of Science for Feb., 1843.
1843. Report of the Committee appointed for the Reduction of Meteorological Observations. From the Report of the British Association for the Advancement of Science for 1843.
1843. A Letter addressed to S. HUNTER CHRISTIE, Esq., Sec. Royal Soc., dated Nov. 15, 1842, on

certain Improvements on Photographic Processes and on Parathermic Rays of the Solar Spectrum. Phil. Trans., Part I. for 1843.

1844. Astron. Nachr., No. 493. Notice of Mr. CLERIHEWS's Observations of the Bifurcation of the tail of the Great Comet of 1843.

1844. Further Remarks on the Revision of the Constellations. Notices of Royal Astron. Soc., Vol. VI. p. 60.

1844. Contributions to Actinometry; also On the Amphitype, a new Photographic Process. British Association Report for 1844.

1844. Astron. Nachr., No. 520. "On M. HOUZEAU's Speculations on the Effect of Aberration in altering the Apparent Periods and Orbits of Double Stars."

1845. Address to the British Association on its Opening at Cambridge on June 19th, 1845. Report of British Association for 1845.

1845. On a Case of Superficial Color presented by a homogeneous Liquid internally colorless. Read Feb. 13th, 1845. Phil. Trans., Part I.

1845. On the Epipölic Dispersion of Light, being a Supplement to a Paper entitled "On a Case of Superficial Color presented by a homogeneous Liquid internally colorless." Read April 3d, 1845. — Phil. Trans., Part I

1845. Sixth Report of the Committee, consisting of Sir J. HERSCHEL, the MASTER OF TRINITY COLLEGE, Cambridge, the DEAN OF ELY, Dr. LLOYD, and Colonel SABINE, appointed to conduct the Co-operation of the British Association in the System of Simultaneous Magnetical and Meteorological Observations. Report of the British Association for the Advancement of Science for 1844.

1845. Memoir of FRANCIS BAILY, Esq. From the Monthly Notices of the Royal Astronomical Society, Vol. VI, November, 1844.

1845. Report of a Committee, consisting of Sir JOHN HERSCHEL, Mr. WHEWELL, and Mr. BAILY (deceased), appointed by the British Association in 1840, for revising the Nomenclature of the Stars. Report of the British Association for the advancement of Science for 1844.

1845. "On a Seed-depositing Dibble," in a letter to the Editor of the Gardener's Chronicle. G. C. 1845, No. 12.

1846. Astron. Nachr., No. 584. Observations of BIELA's double Comet on Jan. 28th, 1846.

1846. Notices of Astron. Soc., Vol. VII. p. 95. Letter to Captain SMYTHE on the change of Magnitude of β Ursæ Minoris.

1846. Letter to Editor on LEVERRIER's (ADAM's) Planet. Athenæum, No. 988, p. 1019.

1846. Letter to Editor of the Guardian on do. do. Guardian, Oct. 28th, 1846.

1846. Letter to Editor of the Times, Oct., 1846.

1847. Obituary Notice of BESSEL. Notices of Astron. Soc., Feb. 12th, 1847.

1847. On the Action of Lime in Preserving liquid Manure. Gardener's Chronicle, 1847, No. 15, p. 244.

1847. Results of Astronomical Observations made during the Years 1834, 5, 6, 7, 8, at the Cape of Good Hope, being the Completion of a Telescopic Survey of the whole Surface of the visible Heavens, commenced in 1825. SMITH & ELDER, London.

1847. Astron. Nachr., No. 616. Extract of a Letter to Prof. SCHUMACHER on the Nomenclature of the small Planets.

1847. Letter to Editor of the Times on the Planet Neptune.

1847. Review of HUMBOLDT's Kosmos. Edinburgh Review, Jan. 1848.

1848. Obituary Notice of Miss CAROLINE LUCRETIA HERSCHEL. Athenæum, No. 1066.

1848. Address to the Royal Astron. Soc. on delivering Testimonials of Merit to Messrs. HANSEN, HENCKE, HIND, BISHOP, LUBBOCK, LEVERRIER, ADAMS, ARGELANDER, AIRY, EVEREST, and WEISSE. Notices, Feb., 1848.

1848. Athenæum, No. 1100, p. 1176. Letter to Editor on the Planet Neptune.

1849. Athenæum, No. 1100. On dividing the Vanishing Lines in Perspective.

1848. Athenæum, No. 1099, p. 1149. Account of a Lunar Rainbow.

1848. On the Effects of the Moon on the Weather. (Lord LOVELACE's Essay on Climate as connected with Husbandry.)

1849. Address delivered on presenting the Honorary Medal of the Royal Astronomical Society to WILLIAM LASSELL, Esq., of Liverpool. Royal Astron. Soc., Vol. IX. No. 4.

1849. On the Determination of the most probable Orbit of a Binary Star. Read April 13th, 1849. Trans. Royal Astron. Soc.

1849. Outlines of Astronomy. LONGMAN, London. (This has been translated into Arabic and Chinese.) N. B. — A 5th Edition much enlarged, published in 1858.

1849. Preface, Article Meteorology, Appendix to Astronomy, and Appendix to Ethnology in the "Naval Officer's Scientific Manual," edited by Sir J. HERSCHEL.

1850. Athenæum for Sat., Feb. 2d, 1850. Latin Epigram — "In cupidinem flantem." Also, Notice of an Error of 2 days left uncorrected in the Gregorian Reformation of the Calendar.

1850. Review of M. QUETELET's Lettres sur la Théorie des Probabilities, and Translation. Edinburgh Review, July, 1850.

1850. On the Algebraic Expression of the Number of Partitions of which a given Number is susceptible. Read May 16th, 1850. Phil. Trans., Part II., for 1850.

1851. Reclamation on Mr KEMP's claiming my Process for Separation of Iron from other Metals. Phil. Magazine, No. 170.

1853. On the Substitution of Bromine for Iodine in Photography. Journal of the Photographic Soc., No. 6, p. 70, June 21st, 1853.

1853. Athenæum, No. 1341, p. 83 Letter to Editor on Mr. J. STEWART's new Photographic Processes.

1854. Evidence on Examination before the Select Committee of the House of Commons on the Decimal Coinage. See Report of Committee, 1853, p. 45–62.

1854. Circular issued to Bankers by Sir J. H., Master of the Mint, proposing Inquiries respecting the Circulating of the Florin. Report of the Committee for Decimal Coinage, 1857, p. 125.

1854. Analysis of Replies of Bankers, &c., to the Queries on the Circulation of the Florin. Preliminary Report of Committee on Decimal Coinage, 1857.

1856. Remarks on Slaty Cleavage and the Contorsian of Rocks, in a Letter to T. TYNDALE, Esq., F. R. S.

1856. Letter to Dr. BOOTH, Chairman to the Society of Arts, &c., on the proposed Examination System of the Society. Literarium and Educational Gazette, March 26th, 1856.

1857. The article "Meteorology," Encyclopædia Britannica, Vol. XIV.

1857. Letter in the "Times" signed "Helioscopus," April, 1857.

1857. A volume entitled "Essays from the Edinburgh and Quarterly Reviews with Addresses and other Pieces." LONGMAN & Co., London, one vol. 8vo.

1857. Replies to Lord OVERSTONE's Queries on the Decimal Coinage. Printed in the Report of the Committee for considering the Subject of the Decimal Coinage, 1857.

1858. Sensorial Vision: a Paper read before the Philosophical and Literary Society of Leeds, Sept. 30th, 1858.

1858. Introductory Address to the Chemical Section of the British Association on taking the Chair of that Section in Leeds, in 1858. Sept 23d.

1858. Speech in the Town Hall at Leeds, on Occasion of presenting the Prizes for the Oxford Competitive Examination to the Successful Candidates. Reported in the Leeds Mercury, Sept. 1858.

1858. Report of the Joint Committee of the Royal Soc., and the British Association for procuring Continuance of the Magnetic and Meteorological Observations.

1858. Speech in the Leeds Commercial Exchange, on the Occasion of presenting Prizes to the Successful Candidates of the Young Men's Christian Institute for the Society of Arts Competitive Examinations.

1858. A Letter to the Editor of the Photographic News, dated Oct. 31st, 1858. Photographic News.

1858. A Letter to the Editor of the Photographic News, dated Nov. 6th, 1858, on the "Stereoscopic Angle."

1858. Obituary Notice of the late Very Rev. Dr. PEACOCK, DEAN OF ELY. From the Proceedings of the Royal Soc. for Nov 30, 1858.

1859. Remarks on Color Blindness. From the Proceedings of the Royal Soc. for May 26th, 1859.

1859. Manual of Meteorology. Extracted from the Admiralty Manual of Scientific Enquiry. Third Edition, 1859.

1859. A Letter to the Chairman of the Musical Pitch Committee on Uniform Musical Pitch, dated June 14th, 1859.

1859. The article "Physical Geography," Encyclopædia Britannica, Vol. XVII.

1859. A Letter to the Editor of the Photographic News, dated August 20th, on the Action of the Solar Spectrum upon certain Compounds of Silver. Photographic News.

1859. On a New Projection of the Sphere. Communicated to the Royal Geographical Society, March, 1859. Read April, 1859. Transactions of R. G. S., 1860–61.

1859. A Letter to the Chairman of the Society of Arts Committee for considering and reporting on the Adoption of a Standard of Musical Pitch. Journal of the Society of Arts, No. 346, Vol. VII., July 8th, 1859.

1859. A third Letter to the Editor of the Photographic News, dated July 16, 17, on the Action of the Solar Spectrum on Iodized, Bromized, and other Papers. Photographic News, July 17th, 1859.

1859. A fourth Letter, dated Aug. 20th, to the Editor of the Photographic News, on the Use of Lactate of Silver, and on the Spectra impressed on Iodinetted, Brominetted, and other Papers, by the Action of the Solar Rays. Photographic News, Sept 9th, 1859, Vol. III. p. 2.

1860. Athenæum, No. 1696, p. 581. British Modular Standard of Length. — Letter to Editor.

1860. Demonstration of Dr. BRINKLEY's Expression of the Coefficients in the Development of an Exponential Function Phil. Trans. Royal Soc. London, 1860.

1860. Athenæum, No. 1697, p. 216. A second Letter to the Editor of Athenæum on the Standard of Length.

1860. A fifth letter to Editor of Photographic News on Representation of Scenes in Action, and On Instantaneous Photography.

1860. Replies to Queries circulated by the Royal Commission for Light-House Buoys and Beacons, printed in the Report of the Commission.
1860. A Letter to the Editor of the "Photographic News," dated June 30th, 1860.
1861. An Article on a Swing Cot for Sea Voyages. From the Journal of the Society of Arts, Jan. 4th, 1861.
1861. The article "Telescope," Encyclopædia Britannica, Vol XXI.
——? Notice of a remarkable Occurrence of Serpentine at the Junction of Sienite with the Dolomite of the Tyrol.

This Catalogue of Titles was prepared by the author at our request. It would have been very difficult, if not impossible, for any other hand to have made it so complete. It is an interesting and valuable record of the incessant labor of half a century: remarkable both for the number of papers included and the variety of subjects treated. We had intended to give in this connection a short biographical sketch of the distinguished author, but want of space forbids. We will, however, refer our readers to a *Biographical Dictionary*, now in course of publication by MESSRS. GRIFFIN of London, for a sketch, which is, so far as it goes, exact, and has received SIR JOHN HERSCHEL'S sanction.

While expressing our sincere thanks for the preparation of this catalogue and for a large number of the memoirs included in it, we would also acknowledge the receipt of the following very valuable papers by the author's illustrious father, SIR WILLIAM HERSCHEL: —

1789. Catalogue of a Second Thousand of New Nebulæ and Clusters of Stars; with a few Introductory Remarks on the Construction of the Heavens. Phil. Trans. Read at the Royal Society, June 11th, 1789.

1796. On the Method of observing the Changes that happen to the Fixed Stars; with some Remarks on the Stability of the Light of our Sun. To which is added a Catalogue of Comparative Brightness, for ascertaining the Permanency of the Lustre of Stars. Read before the Royal Society, Feb. 25th, 1796. Phil. Trans.

1796. On the Periodical Star α Herculis; with Remarks tending to establish the Rotatory Motion of the Stars on their Axes. To which is added a Second Catalogue of the Comparative Brightness of the Stars. Read before the Royal Society, June 9th, 1796. Phil. Trans.

1797. A Third Catalogue of the Comparative Brightness of the Stars; with an Introductory Account of an Index to Mr. FLAMSTEED'S Observations on the Fixed Stars, contained in the Second Volume of the Historia Cœlestis. To which are added several useful Results derived from that Index. Read before the Royal Society, May 18th. Phil. Trans.

1799. A Fourth Catalogue of the Comparative Brightness of the Stars. Read before the Royal Society, February 21st. Phil. Trans.

1802. Catalogue of Five Hundred New Nebulæ, Nebulous Stars, Planetary Nebulæ, and Clusters of Stars; with Remarks on the Construction of the Heavens. Read before the Royal Society, July 1st, 1802. Phil. Trans.

1803. Account of the Changes that have happened, during the last twenty-five years, in the Relative Situation of Double Stars; and an Investigation of the Cause to which they are owing. Read before the Royal Society, June 9th. Phil. Trans.

Editorial Items.

SIR JOHN HERSCHEL'S PORTRAIT. — It gives us sincere pleasure to be able to present to our readers an excellent likeness of Sir JOHN HERSCHEL. It was engraved by the distinguished artist, H. WRIGHT SMITH, Esq. of Boston, from a Photograph in the possession of Miss MARIA MITCHELL, of Nantucket, who says the likeness is as good as it can be. Miss MITCHELL has kindly allowed us the perusal of an extract of a letter from Lady HERSCHEL, in which a decided preference is given to this Photograph.

We can assure our friends that the artist has been remarkably successful in reproducing the likeness; and it is with the deepest satisfaction that we present it to the numerous admirers of the distinguished original on this side of the Atlantic, by whom we feel assured it will be received with the warmest welcome.

The following gentlemen have sent solutions of the problems in the December number: —

H. J. MCMURPHY, Londonderry, N. H.; ASHER B. EVANS, Madison University, Hamilton, N. Y.; W. L. MARCY, Coventry, N. Y.; J. M. TERRILL, Linneus, Me.; G. B. HICKS, Cleveland, Ohio; EDWIN A. DODDS, Gouverneur, N. Y.; WILLIAM TIMPSON, White Plains, N. Y.; JAMES CLARK, Wayne, Maine; E. S. STEARNS, Chester, N. J.; DAVID TROWBRIDGE, Perry City, N. Y., W. W. LEMMON, M'Kendree College, Lebanon, Ill.; W. G. N., Philadelphia, Pa.; H. P. HENDERSON, N. Y. City.

PRIZES FOR SOLUTIONS. — Those entitled to these Prizes will please address Messrs. SEVER & FRANCIS, stating the number of the Monthly in which the prize is awarded, its amount, the books desired, and how they should be sent. It must not be forgotten that the postage on books must be *prepaid*.

It would add very much to the interest and value of this department of the Monthly if the competition could be largely increased. It is not certain that the good accomplished at present warrants the expense, labor, and space in the Monthly devoted to these prizes.

BOOKS RECEIVED. — A Treatise on the Differential and Integral Calculus. By A. E. CHURCH, LL. D., Professor of Mathematics in the U. S. Military Academy, West Point. A new edition. Published by A. S. Barnes & Burr, 51 and 53 John Street, N. Y. (Notice in next number.)

An Elementary Treatise on the Dynamics of a System of Rigid Bodies. With numerous Examples. By EDWARD JOHN ROUTH, M. A., Fellow and Assistant Tutor of St. Peter's College, Cambridge, Examiner in the University of London. Cambridge: Macmillan & Co.; and 23 Henrietta Street, Covent Garden, London. 1860.

THE

MATHEMATICAL MONTHLY.

Vol. III. . . . MAY, 1861. . . . No. VIII.

PRIZE PROBLEMS FOR STUDENTS.*

I. Let $ABCD$ be a rectangular parallelogram, G, H, two points in AD and DC, and E, F, two points in AB and BC respectively, such that GH equals EF. Join AH, AF, CG, CE, and show that $AH^2 + CG^2 = AF^2 + CE^2$.

II. ABC is an isosceles triangle, AB the base, D a point in a straight line parallel to AB, DE perpendicular to AB; prove that the difference of the triangles ADC, AEC, is a constant area.

III. If in the continued fraction

$$\sqrt{n} = a + \cfrac{1}{b + \cfrac{1}{2c + \cfrac{1}{b + \sqrt{n}}}}$$

we put $a + \frac{1}{b} = M$, $a + \cfrac{1}{b + \frac{1}{c}} = N$, then $MN = n$.

IV. Let

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1, \quad lx + my + nz = 0,$$

be the equations of an ellipse in space; prove that the principal axes of this ellipse are the roots of the equation

$$\frac{l^2 a^2}{z^2 - a^2} + \frac{m^2 b^2}{z^2 - b^2} + \frac{n^2 c^2}{z^2 - c^2} = 0.$$

* The right-hand member of equation 1, Prob. I., March No., should be y^2, instead of $4y^2$.

V. One of two homofocal ellipses is inscribed in a triangle, and the other circumscribed about it; show that the following relation exists between the semi-transverse axes and eccentricity

$$e^8 - 4e^6a^2 + 6e^4a^2a'^2 - 4e^2a^2a'^4 + a^4a'^4 = 0.$$

Solutions of these problems must be received by July 1, 1861.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS IN No. IV., Vol. III.*

The first prize is awarded conjointly to G. B. Hicks, Cleveland, Ohio, and David Trowbridge, Perry City, N. Y., their solutions being considered of equal merit.

The second prize is awarded to John Q. Hollister, Hamilton College, Clinton, N. Y.

The third prize is awarded to George W. Pierce, Freshman Class, Harvard College.

The fourth prize is awarded to Marcus M. Rhoades, Mount Pleasant College, Huntsville, Mo.

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* To prevent misapprehension, we will state here, that it was never intended that any competitor should draw the same prize twice during the same year, although credit will be given him as being otherwise entitled to it; and no prize will be awarded for solutions which do not in themselves possess sufficient merit to fairly entitle them to it. It is a matter of extreme satisfaction that a few of the competitors always send excellent, and often elegant solutions of all the problems, and are certainly preparing themselves for important and responsible positions demanding mathematical attainments and ability. We sincerely hope to witness a great increase in competition for these prizes, and to this end we now remove the restriction limiting the competition to students in the United States, and invite all students interested in this department of the Monthly to send us their solutions. To the inquiry of our young English friends, we reply, that we shall be most happy to have them compete for and carry off these prizes. Your letters sent to us, *post-paid*, at *Cambridge, Mass., U. S. A.*, will be pretty sure to be promptly received. — Ed.

Prize Solution of Problem I.

By W. W. Lemmon, M'Kendree College, Lebanon, Ill.

$ABCD$ is a square, AC its diagonal; bisect AD in E, join BE cutting AC in F; then will

$$\frac{\triangle AEF}{1} = \frac{\triangle CEF}{2} = \frac{\triangle ABE}{3} = \frac{\triangle BCF}{4}.$$

Bisect AB in G and join FG. It is evident that

$$AEF = AFG = GFB. \quad \therefore AEF = \tfrac{1}{2} AFB = \tfrac{1}{3} AEB.$$

But $AEC = AEB$; subtract AEF from each, and $CEF = AFB$ The triangles AEF, BFC are similar; therefore

$$\frac{AEF}{BFC} = \frac{AE^2}{BC^2} = \frac{1}{4}. \qquad \therefore AEF = \tfrac{1}{4} BFC.$$

Prize Solution of Problem II.

By several of the Competitors.

If $\frac{a-b}{ay+bx} = \frac{b-c}{bz+cx} = \frac{c-a}{cy+az} = \frac{a+b+c}{ax+by+cz}$, then each of these fractions $= \frac{1}{x+y+z}$.

Since these ratios are equal, any antecedent is to its consequent as the sum of all the antecedents is to the sum of all the consequents; therefore either fraction equals the sum of all the numerators *divided by the sum of all* the denominators,

$$\therefore \frac{a+b+c}{(a+b+c)(x+y+z)} = \frac{1}{x+y+z}.$$

Prize Solution of Problem III.

If d be the perpendicular, dropped from the angle C on the opposite side of a triangle, show that $\sin C = \frac{cd}{ab}$. Deduce the formula

$$\sin (A+B) = \sin A \cos B + \cos A \sin B.$$

From the vertex B drop a perpendicular d' on the opposite side. Then $a \sin C = d'$, and calling the area of ABC, A, we get

$$ab \sin C = bd' = 2A = cd; \quad \therefore \sin C = \frac{cd}{ab}.$$

Now, $c = a\cos B + b\cos A, \qquad d = b\sin A = a\sin B;$

$$\therefore \frac{cd}{ab} = \sin A\cos B + \cos A\sin B = \sin C = \sin(A+B),$$

since $A + B + C = 180°$. This is substantially the solution given by all the competitors.

Prize Solution of Problem IV.

An hyperbola, having its asymptotes parallel to the axes of co-ordinates, may be made to pass through the angular points of the quadrilateral formed by the four lines whose equations are

(1) $y + x + 6a = 0,$ (3) $y + 6x + a = 0,$
(2) $y + 2x + 3a = 0,$ (4) $y + 3x + 2a = 0.$

Denote the equations (1), (2), (3), (4) by $\alpha, \beta, \gamma, \delta$; then

$$(1) \qquad \alpha\gamma + k\beta\delta = 0$$

is the equation of a conic passing through the angular points of the quadrilateral $(\alpha\beta\gamma\delta)$; for it is satisfied by $\alpha = 0, \beta = 0$; $\alpha = 0, \delta = 0$; $\gamma = 0, \beta = 0$; $\gamma = 0, \delta = 0$; k being an arbitrary constant. We have then

$$(2)\ y^2 + 7xy + 6x^2 + 7ay + 37ax + 6a^2 + k(y^2 + 5xy + 6x^2 + 5ay + 13ax + 6a^2) = 0.$$

To remove the second powers of x and y make $k = -1$,

$$(3) \qquad \therefore xy + ay + 12ax = 0.$$

Put $x = x' + b, y = y' + c'$, then (3) becomes

$$(4)\quad x'y' + (c + 12a)x' + (a + b)y' + bc + ac + 12ab = 0;$$

and by making $c = -12a$, $b = -a$, we get $x'y' = 12a^2$, the well-known equation of the hyperbola referred to its asymptotes. Therefore (3), which passes through the angular points of the quadrilateral, is an hyperbola having its axis of co-ordinates parallel to its asymptotes. This solution was given by Messrs. Hicks and Trowbridge.

Mr. ASHER B. EVANS gave the following solution: —

Suppose the centre of the hyperbola to be at the point (θ, φ), and remove the origin of coördinates to this point; then will equations, (1), (2), (3), (4) become

(5) $y+x+6a-\theta-\varphi=0$, (7) $y+6x+a-6\theta-\varphi=0$,
(6) $y+2x+3a-2\theta-\varphi=0$, (8) $y+3x+2a-3\theta-\varphi=0$.

Let A, B, C, D represent the intersection of (1) and (2), (1) and (4), (2) and (3), (3) and (4), in forming the quadrilateral $ABCD$. From equations (5), (6), (7), (8), we readily find

$$A=(\theta+3a, \varphi-9a), \qquad B=(\theta+2a, \varphi-8a),$$
$$C=(\theta+\tfrac{1}{2}a, \varphi-4a), \qquad D=(\theta+\tfrac{1}{3}a, \varphi-3a).$$

The equation to the hyperbola, referred to its asymptotes as axes, is

(9) $$xy=c^2.$$

If the hyperbola pass through A, B, C, we must have

(10) $$xy=c^2=(\theta+3a)(\varphi-9a)=(\theta+2a)(\varphi-8a)$$
$$=(\theta+\tfrac{1}{2}a)(\varphi-4a).$$

Equation (10) gives $\theta=a$, $\varphi=12a$; hence

(11) $$xy=c^2=(\theta+3a)(\varphi-9a)=(\theta+2a)(\varphi-8a)$$
$$=(\theta+\tfrac{1}{2}a)(\varphi-4a)=(\theta+\tfrac{1}{3}a)(\varphi-3a)=12a^2.$$

Since equation (11) exists, the hyperbola which passes through A, B, C, also passes through D, and hence through all the angular points of the quadrilateral $ABCD$.

PRIZE SOLUTION OF PROBLEM V.

By DAVID TROWBRIDGE, Perry City, N. Y.

Three lines, x, y, z, are drawn from the angular points of a triangle to a point within it. Show that when $x^n+y^n+z^n$ is a minimum we have

$$\frac{\sin(x, y)}{z^{n-1}}=\frac{\sin(x, z)}{y^{n-1}}=\frac{\sin(y, z)}{x^{n-1}},$$

in which (x, y) denotes the angle between x and y.

Let ABC be the triangle, and O the point within it, $OC = x$, $OB = y$, $OA = z$, and the angle $(x, a) = \theta$. Then we readily get

$$u = x^n + y^n + z^n,$$

$$= x^n + (a^2 + x^2 - 2ax\cos\theta)^{\frac{n}{2}} + (b^2 + x^2 - 2bx\cos(C-\theta))^{\frac{n}{2}},$$

$$D_x u = nx^{n-1} + ny^{n-2}(x - a\cos\theta) + nz^{n-2}(x - b\cos(C-\theta)) = 0,$$

$$(1) \qquad = x^{n-1} + xy^{n-2} + xz^{n-2} - ay^{n-2}\cos\theta - bz^{n-2}\cos(C-\theta) = 0,$$

$$D_\theta u = nax y^{n-2}\sin\theta - nbx z^{n-2}\sin(C-\theta) = 0,$$

$$(2) \qquad = ay^{n-2}\sin\theta - bz^{n-2}\sin(C-\theta) = 0.$$

By putting $B = x^{n-2} + y^{n-2} + z^{n-2}$, (1) becomes

$$(3) \qquad Bx = ay^{n-2}\cos\theta + bz^{n-2}\cos(C-\theta).$$

Multiplying (3) by $\sin\theta$, and (2) by $-\cos\theta$, and adding the results, we get

$$Bx\sin\theta = bz^{n-2}\sin C, \qquad \text{or} \qquad Bax\sin\theta = abz^{n-2}\sin C.$$

But $ax\sin\theta = xy\sin(x, y) =$ twice the area of BOC;

$$(4) \quad \therefore \frac{Bxyz\sin(x, y)}{z^{n-1}} = ab\sin C = \text{twice the area of the triangle } ABC.$$

Similarly,

$$\frac{Bxyz\sin(x, z)}{y^{n-1}} = ac\sin B, \qquad \frac{Bxyz\sin(y, z)}{x^{n-1}} = bc\sin A;$$

$$\therefore \frac{\sin(x, y)}{z^{n-1}} = \frac{\sin(x, z)}{y^{n-1}} = \frac{\sin(y, z)}{x^{n-1}}.$$

The condition of a minimum

$$D_x^2 u \,.\, D_\theta^2 u - (D_{x,\theta}^2 u)^2 > 0.$$

is readily computed.

Mr. G. B. Hicks computed the values of x, y, z in a very elegant manner for $n = 1$, and $n = 2$, which we reluctantly omit for want of space.

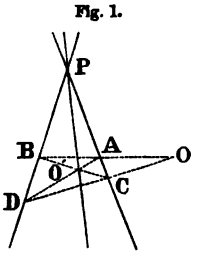
ON THE GEOMETRICAL CONSTRUCTION OF CERTAIN CURVES BY POINTS.

By H. A. Newton, Professor of Mathematics in Yale College.

1. Definition. — *The polar to a poim with respect to an angle* is that straight line which, together with the line drawn from the point to the vertex of the angle, divides the angle harmonically. (*Géom. Sup.*, Art. 347.)*

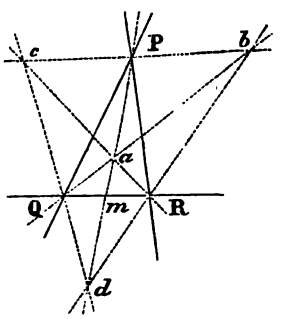
2. It may be constructed by the straight edge alone. For let O (Fig. 1) be the given point, and APB the given angle. Draw OAB and OCD, any two straight lines, through O to cut PA in A and C, and PB in B and D; draw AD and BC to cut each other in O', and draw PO'. The line PO' will be the polar of the point O with respect to the angle APB. (*Géom. Sup.*, Art. 346.)

Fig. 1.



3. Theorem. — Let PQR (Fig. 2) be any triangle, and a any point in the same plane. Draw aP, aQ, aR, and also Pb, Qc, and Rd, the polars to a with respect to the three angles of the triangle PQR. Then the lines Pa, Qc, and Rd will meet in one point, d, the lines Pb, Qa, and Rd will meet in one point, b, and Pb, Qc, and Ra will meet in one point, c.

Fig. 2.



For let Pa meet QR, Qc, and Rd in m, d', and d respectively. The segment Pm is cut harmonically by Qa and Qd', and also by Ra and Rd. Therefore d and d' coincide; or, in other words, Pa, Qc, and Rd meet in one point. In like manner the other lines may

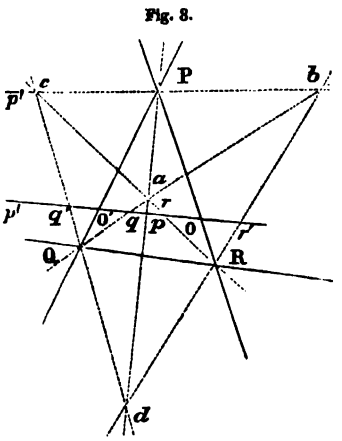
* The references are to the *Traité de Géométrie Supérieure*, par M. Chasles, 8vo, Paris, 1852, Bachelier. We cannot too highly recommend this work of the great French Geometer.

be proved to meet in b, and in c. This proposition is proved also in the *Géom. Sup.*, Art. 373.

4. In the figure thus constructed the six lines joining the four points a, b, c, d may be considered the four sides and two diagonals, or in other words the six sides of a complete quadrilateral, $a\,b\,c\,d$. The points P, Q, and R will be the points of intersection of the three couples of opposite sides.

5. THEOREM. — The three polars of any point, O (Fig. 3), with respect to the three angles formed at P, Q, and R by the opposite sides of the quadrilateral meet in one point, O'.

Fig. 3.



This proposition is demonstrated by CHASLES (*Géom. Sup.*, Art. 347), as follows: Let the polars of O, with respect to the angles $a\,P\,b$ and $a\,Q\,c$, meet in O'; it is to be proved that the line joining O' and R is polar to O, with respect to the angle $a\,R\,d$. Draw $O\,O'$, and let this line meet the sides of the quadrilateral in p, p'; q, q'; r, r'. These six points are in involution (*Géom. Sup.*, Art. 339); and since the segment $O\,O'$ divides harmonically the segments $p\,p'$ and $q\,q'$, the points O and O' are the double points of the involution (*Géom. Sup.*, Art. 205). Hence the segment $r\,r'$ divides the segment $O\,O'$ harmonically, and O' is on the polar of O with respect to the angle $a\,R\,d$.

6. Since there is but one polar to a point with respect to an angle, it is evident that, having constructed the quadrilateral (or having assumed the point a), there is *in general* to one point, O, of the plane $P\,Q\,R$ one point, O', and only one, where the polars of O intersect.

These two points, O and O', have important relations to each other; and in what follows, each will be said to be *conjugate* to the other.

7. If O is on one of the sides of the quadrilateral, as $a\,d$, then,

1st, O' is on the same line $a\,d$; and

2d, the segment, $O\,O'$, divides $a\,d$ harmonically; for,

1st, the polar of O, with respect to the angle $a\,P\,b$, is the side of the angle $P\,a$; and

2d, the lines $O\,R$ and $O'\,R$ must divide the angle $a\,R\,d$, and hence the segment $a\,d$, harmonically.

8. A fasceau of conics circumscribing the quadrilateral $a\,d\,Q\,R$ will, by a well-known theorem, cut the line $b\,c$ in a series of couples of points in involution. And since the two lines $b\,a$, $b\,d$ may be considered as one of these conics, the point b is one of the double points of the involution. The point c, for a like reason, is the other double point. Therefore each segment, as $m\,n$, intercepted on the line $b\,c$ by a conic about $a\,d\,Q\,R$, is divided harmonically by b and c (*Géom. Sup.*, Art. 205). Therefore m and n are points conjugate to each other (Art. 7). Hence, if a conic be described to pass through a and d, two summits of the quadrilateral, and through Q and R, those two angles of the triangle not on the line $a\,d$, then will m and n, the two points where this conic cuts the opposite side, $b\,c$, of the quadrilateral, be conjugate to each other.

9. The point a is entirely arbitrary. The positions of conjugate points may, however, be more readily conceived if a is taken the centre of the circle inscribed in the triangle $P\,Q\,R$. Then will the lines $a\,P$, $a\,Q$, $a\,R$ bisect the angles of the triangle, the lines $P\,b$, $Q\,c$, and $R\,d$ will bisect the exterior angles of the triangle, and be perpendicular respectively to the lines $a\,P$, $a\,Q$, and $a\,R$. The points b, c, and d will be the centres of the circles touching the sides of the triangle $P\,Q\,R$, and the angles $a\,P\,O$, $a\,Q\,O$, and $a\,R\,O$ will be equal respectively to $a\,P\,O'$, $a\,Q\,O'$, and $a\,R\,O'$.

The centre of a circle inscribed in a given triangle cannot, however, be determined by a straight edge alone; and hence, in con-

structions where the compasses are not to be used, the point a must not be subject to the above condition.

10. Definitions. — If the point O move along any curve, its conjugate point O' will describe a second curve. This second curve will be called *the conjugate* of the first curve. The first curve is evidently the conjugate of the second.

The points P, Q, and R will be called *base points*. The three lines $P\,Q$, $P\,R$, and $Q\,R$ will be called *axes*.

11. If two curves intersect in n points, none of which are base points, the two conjugate curves will evidently intersect in n points, none of which are base points.

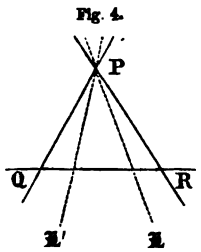
12. If two curves touch each other at a point not a base point, the two conjugate curves will also touch each other.

For two curves, $\mathfrak{A}$ and $\mathfrak{B}$, are tangent to each other in a point O, when the segment intercepted by them on any transversal indefinitely near to O is an infinitesimal of the second order. Let the straight line $P\,m\,n$, indefinitely near to O, cut $\mathfrak{A}$ and $\mathfrak{B}$ in m and n, and let m', n', and O' be the points conjugate to m, n, and O, and therefore on the curves conjugate to $\mathfrak{A}$ and $\mathfrak{B}$. The points, m', n', and P will evidently be in a straight line, which will be indefinitely near to the line $O'\,P$. The angle made by the lines $m\,Q$ and $n\,Q$, and hence the angle made by $m'\,Q$ and $n'\,Q$ will be infinitesimals of the second order. Therefore, if the directions of $m'\,P$ and $m'\,Q$ make a finite angle, the segment $m'\,n'$ is an infinitesimal of the second order, and the curves conjugate to $\mathfrak{A}$ and $\mathfrak{B}$ are tangent to each other at O'.

13. When the point O' is on an axis, as $P\,Q$, or is at infinity, this demonstration fails. In the first case, the point O is the base point R. It is easy to see that if two curves intersect in an axis, their conjugate curves touch each other in the opposite base point. This is only an apparent exception to the proposition of Art. 11th, for

the two curves tangent at the base point may be considered as intersecting each other in the base point and in a second point indefinitely near to the base point.

14. The curve conjugate to a straight line, $\mathfrak{L}$, passing through one, and only one, of the base points, as P, is a straight line passing through the same base point, P.



For let $\mathfrak{L}$ (Fig. 4) be drawn so that $\mathfrak{L}$ and $\mathfrak{L}'$ shall divide harmonically the angle made at P by the opposite sides of the quadrilateral. Then if O moves along $\mathfrak{L}$, the point O' must move along $\mathfrak{L}'$; that is, if $\mathfrak{L}$ and $\mathfrak{L}'$ are conjugate to each other.

15. If the line $\mathfrak{L}$ is an axis, there is no conjugate line.

For since $P\,Q$ and $P\,R$ divide harmonically the angle made at P by the opposite sides of the quadrilateral, the point conjugate to any point on $P\,Q$ is on $P\,R$. For a like reason the point conjugate to any point on $P\,Q$ is on $Q\,R$. Hence the point of intersection, R, is to be considered the conjugate of every point in $P\,Q$.

It was stated above (Art. 6), that *in general* each point, O, of the plane, $P\,Q\,R$, had one, and only one, conjugate point. The base points, P, Q, and R, are the only exceptions.

16. The curve conjugate to a straight line, not passing through a base point, is a conic that passes through the three base points.

For let a, b, c, d, &c., be points on the straight line, and a', b', c', d', &c., be the points respectively conjugate to them, and let lines be drawn from P and Q to the several points. Then the couples of lines $P\,a$, $P\,a'$; $P\,b$, $P\,b'$; $P\,c$, $P\,c'$, &c., are in involution (*Géom. Sup.*, Art. 244), and hence the anharmonic function of the lines $P\,a'$, $P\,b'$, $P\,c'$, and $P\,d'$ is equal to that of $P\,a$, $P\,b$, $P\,c$, and $P\,d$ (*Géom. Sup.*, Arts. 236 and 13); which is equal to that of $Q\,a$, $Q\,b$, $Q\,c$, and $Q\,d$; which is equal to that of $Q\,a'$, $Q\,b'$, $Q\,c'$, and $Q\,d'$. Therefore the locus of the points a', b', c', d', &c., is a conic passing through P and Q (*Géom. Sup.*, Art. 542), and hence by symmetry through R.

17. Conversely, the conjugate to a conic passing through the three base points, is a straight line that does not pass through a base point.

18. DEFINITIONS. — A curve of the nth degree is one that is cut by every straight line in its plane in n points (real and imaginary), and in n points only.

A *conic* is a curve of the second degree.

A *cubic* is a curve of the third degree.

SALMON proposed that the term cubic, used in Algebra to denote an equation of the third degree, should be used in Geometry to denote a curve of the third degree. As we shall have occasion frequently to speak of curves of the fourth degree, we shall make a similar use of the term biquadratic, by which we shall mean a curve of the fourth degree.

19. The curve conjugate to a conic $\mathfrak{C}$ that passes through two, and only two, of the base points, as P and Q, is a conic that passes through the same two base points, and does not pass through the third base point.

It is a conic. For any conic through P, Q, and R cuts the conic $\mathfrak{C}$ in four points; that is, in two points other than the base points, P and Q. Therefore any straight line (Art. 17) cuts the conjugate to $\mathfrak{C}$ in two, and only two, points (real or imaginary), namely, the two points conjugate to the intersections of the first two conics.

Moreover, the conic $\mathfrak{C}$ crosses the axis $P\,R$ in one point, D, other than P. The points conjugate to points indefinitely near to D will evidently be on the line through Q, conjugate to $D\,Q$, and these points will be indefinitely near to Q; in other words, the conjugate to $\mathfrak{C}$ passes through Q, and has for its tangent the line conjugate to $D\,Q$. For a like reason it passes through P. It does not, however, pass through R; for its conjugate, that is, $\mathfrak{C}$, would then be a straight line (Art. 17).

20. The curve conjugate to a conic $\mathfrak{C}$ through one, and only one, base point, P, is a cubic that passes through the other two base points, and has a double point at P.

It is a cubic. For any conic through P, Q, and R cuts $\mathfrak{C}$ in three, and only three, points (real and imaginary) other than P. Hence any straight line cuts the conjugate to $\mathfrak{C}$ in three, and only three, points.

Again, since $\mathfrak{C}$ cuts PQ and PR each in one point other than P, the conjugate to $\mathfrak{C}$ passes through the points R and Q. And since $\mathfrak{C}$ crosses QR in two points, neither of which is a base point, there are two directions in which the conjugate to $\mathfrak{C}$ passes through P; that is, there are two tangents (real or imaginary) to the curve at P, and hence P is a double point. If the tangents are imaginary, then P is an isolated point.

21. The curves conjugate to a conic, $\mathfrak{C}$, that does not pass through any base point, is a biquadratic, having double points at each base point.

It is a biquadratic. For any conic through P, Q, and R cuts $\mathfrak{C}$ in four points (real or imaginary), neither of which is a base point. Hence any straight line cuts the conjugate to $\mathfrak{C}$ in four points (real or imaginary).

Again, $\mathfrak{C}$ cuts each axis in two points, neither of which is a base point. Hence the conjugate to $\mathfrak{C}$ has two tangents (real or imaginary) at each base point.

22. These are single examples of the following general theorem. Let $\mathfrak{A}$ be a curve of the nth order, having a multiple point of the p order at P, a multiple point of the q order at Q, and a multiple point of the r order at R, and let $\mathfrak{A}'$ be conjugate to $\mathfrak{A}$, then,

1st, $\mathfrak{A}'$ is a curve of the $(2n-p-q-r)$th degree; and

2d, It will have at P a multiple point of the $(n-q-r)$th order, at Q a multiple point of the $(n-p-r)$th order, and at R a multiple point of the $(n-p-q)$th order.

1st. $\mathfrak{A}'$ is a curve of the $(2n-p-q-r)$th order.

For any conic cuts $\mathfrak{A}$ in $2n$ points. If the conic passes through P, Q, and R, it will cut $\mathfrak{A}$ in those points respectively, p, q, and r times; and therefore in $(2n-p-q-r)$ other points. Hence, any straight line will cut $\mathfrak{A}'$ in $(2n-p-q-r)$ points.

2d. Since $\mathfrak{A}$ crosses QR in $(n-q-r)$ points other than Q and R, $\mathfrak{A}'$ has $(n-q-r)$ branches passing through P. For a like reason $\mathfrak{A}'$ has $(n-p-r)$ branches at Q, and $(n-p-q)$ branches at R.

These branches may, of course, be imaginary by couples, and when $\mathfrak{A}$ does not pass through P we must, in the above theorem, make $p=o$, and so of the rest.

23. To illustrate the above proposition, let $\mathfrak{A}$ be a cubic.

(a) If $\mathfrak{A}$ passes through neither base point, $\mathfrak{A}'$ is a curve of the sixth degree, having triple points at P, Q, and R.

(b) If $\mathfrak{A}$ passes through only one base point, P, $\mathfrak{A}'$ is a curve of the fifth degree, having a triple point at P, and double points at Q and R.

(c) If $\mathfrak{A}$ passes through P and Q, and not through R, $\mathfrak{A}'$ is a biquadratic having double points at P and Q, and a single point at R.

(d) If $\mathfrak{A}$ passes through P, Q, and R, $\mathfrak{A}'$ will be a cubic passing through P, Q, and R.

(e) If $\mathfrak{A}$ has a double point at P, and does not pass through Q nor R, $\mathfrak{A}'$ is a biquadratic having a triple point at P, and single points at Q and R.

(f) If $\mathfrak{A}$ has a double point at P and a single point at Q, $\mathfrak{A}'$ is a cubic having a double point at P, and single point at Q.

(g) If $\mathfrak{A}$ has a double point at P, and passes through Q and R, $\mathfrak{A}'$ is a conic passing through P, but not through Q nor R.

24. If it be required to construct by points a curve that shall satisfy certain conditions, it may be frequently accomplished by con-

structing points of a conjugate curve that shall satisfy certain corresponding conditions, and then finding the points conjugate to those constructed. This method of construction is especially applicable when the curve is to pass through certain points, or touch certain curves.

25. Thus suppose it was required to construct other points of the conic that passes through five given points.

Take three of the given points for base points, and find the points conjugate to the remaining two points. Then draw through these a straight line, and construct the conjugates to any points on this line. They will evidently be on the conic passing through the five given points (Art. 16).

26. Again, to construct by points a cubic having given one double and six single points of the curve

Take the given double point and two of the single points as base points, and construct the points conjugate to the remaining four points. Through the double point and these four conjugate points construct by points (Art. 25) a conic. The conjugate to the points of this conic will be points of the required cubic (Art. 20).

27. To construct a biquadratic having given one triple point and eight single points.

Take the triple point and two single points as base points, and construct the points conjugate to the remaining six points. Through these construct (Art. 26) a cubic having a double point at the given triple point, and then construct points conjugate to points of this cubic. They will be on the required curve (Art. 23 *e*).

28. To construct a curve of the 5th degree having given one quadruple and 10 single points.

With the quadruple point and two single points as base points, construct the conjugate to the remaining eight points. Through these construct (Art. 27) a biquadratic having the given quadruple

point for a triple point, and construct points conjugate to points of this biquadratic. They will be on the required curve.

29. In like manner may be constructed the curve of the sixth degree having given a multiple point of the 5th order and twelve single points.

And in general by repeating the same process may be constructed a curve of the nth degree having a given multiple point of the $(n-1)$st order, and $2n$ single points.

Moreover, since the conjugate of any point may be constructed by the straight-edge alone (Art. 2), all the preceding constructions may be performed without the aid of the compasses.

30. To construct the curve of the $2n$th degree having given three points of the nth order, one point of the $(n-1)$st order, and $2n$ single points.

Take the three points of the nth order as base points, and the construction depends upon that of Art. 29.

31. In like manner may we construct the curve of the $(2n+1)$st degree having given one point of the $(n+1)$st order, three points of the $(n-1)$st order, and $(2n+1)$ single points.

32. The construction of a cubic that shall pass through nine points is given by CHASLES (*Comptes Rendus*, Tom. XXXVI., Séance du 30 Mai, 1853). Hence, to construct the curve of the nth degree having given a point of the $(n-2)$d order, $(n-3)$ double points and eight single points; take the point of the $(n-2)$d order, a double point, and a single point, as base points, and the construction is made to depend upon that of the curve of the $(n-1)$st degree having a point of the $(n-3)$d order, $(n-4)$ double points and eight single points; that is, on a similar problem to the one required, and of the next inferior degree. Repeating the process, the construction is made to depend at last on the construction of the cubic through nine given points.

[To be continued.]

SECOND PRIZE ESSAY. — ON CENTRAL AND DISTURBING FORCES.*

By DAVID TROWBRIDGE, Perry City, N. Y.

25. ASSUME the *elements* of the planetary orbits to be known: $\pi =$ the longitude of the perihelion, $\Omega =$ the longitude of the ascending node, $i =$ the inclination. Put $\varphi =$ the true, and $E =$ the eccentric, and $M =$ the mean anomaly; $\lambda =$ the true longitude, and $l =$ the true latitude. All these quantities are regarded as heliocentric.

The equations of an ellipse, in terms of the eccentric anomaly, are,

$$x = a(\cos E - e), \qquad y = b \sin E. \tag{71}$$

Whence we obtain

$$\sqrt{(x^2 + y^2)} = r = a(1 - e \cos E). \tag{72}$$

We also have

$$r = \frac{a(1-e^2)}{1 + e \cos \varphi}. \tag{73}$$

Equating these two values of r, and reducing, we find

$$\tan \tfrac{1}{2} \varphi = \sqrt{\frac{1+e}{1-e}} \cdot \tan \tfrac{1}{2} E. \tag{74}$$

The differential coefficients of (71), with respect to the time, are,

$$D_t x = -a \sin E D_t E, \qquad D_t y = b \cos E D_t E. \tag{75}$$

If we change the axis from which the area is reckoned (Art. 5), equation (12) gives

$$c = x D_t y - y D_t x. \tag{76}$$

Substituting in (76) from (71) and (75), we find

* This Essay is a continuation of the one in Vol. II. p. 150, of the Mathematical Monthly.

$$(77) \qquad c = \frac{2\pi ab}{T} = ab(1 - e\cos E)\, D_t E.$$

If we put $\frac{2\pi}{T} = n$, reduce, and integrate, we have

$$(78) \qquad nt = M = E - e\sin E,$$

since n is the angle described in the unit of time.

To find E in terms of M, we have

$$(78) \quad E = M + e\sin E, \quad \text{and} \quad \sin E = \sin[M + e\sin E].$$

The continual substitution of this value of $\sin E$ in (79) gives the series

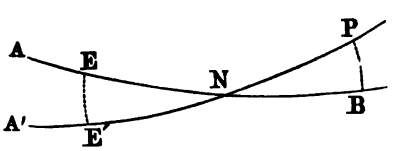
$$(80) \quad E = M + e\sin[{}_1M + e\sin[{}_2M + e\sin[{}_3M + \ldots + e\sin[{}_\infty M].$$

If e is small, a few terms only need be employed.

26. From equations (1), (2), (3), (41), and (20) we have

$$(81) \quad D_t^2 x = -\frac{mx}{r^3}, \quad D_t^2 y = -\frac{my}{r^3}, \quad D_t^2 z = -\frac{mz}{r^3}, \quad D_t \varphi = \frac{c}{r^2}.$$

Before we integrate these equations it will be necessary to transform them a little. For this purpose, let ANB be the ecliptic, $A'NP$ the orbit of the planet, N the ascending node, A the Vernal Equinox, E' the perihelion, and P the planet. Then $AN = \Omega$, $AE = \pi$, and $A'N = \Omega'$, $A'E' = \pi'$, the orbit values of the same quantities; $ANA' = i = PNB$, $E'P = \varphi$, $AB = \lambda$, $PB = l$. Put $NP = \beta$, $NB = \beta'$. We then have, by making $E'N = \omega' = \Omega' - \pi'$,



$$(82) \quad \varphi = \beta + E'N = \beta + A'N - A'E' = \beta + \Omega' - \pi' = \beta + \omega'.$$

Differentiating with respect to t, we have

$$(83) \qquad D_t \varphi = D_t \beta;$$

and hence (81) gives $\qquad D_t \beta = \frac{c}{r^2}$.

By Analytical Geometry we have

$$(84) \qquad x = r\cos l\cos\beta', \qquad y = r\cos l\sin\beta', \qquad z = r\sin l.$$

Since PNB is a right spherical triangle, Spherical Trigonometry gives

$$(85) \qquad x = r\cos\beta, \qquad y = r\cos i\sin\beta, \qquad z = r\sin i\sin\beta.$$

(83) and (85) being substituted in (81), give

$$(86) \quad D_t^2 x = -\frac{m}{c}\cos\beta\, D_t\beta, \qquad D_t^2 y = -\frac{m}{c}\cos i\sin\beta\, D_t\beta,$$
$$D_t^2 z = -\frac{m}{c}\sin i\sin\beta\, D_t\beta.$$

Integrating, and we obtain

$$(87) \qquad D_t x = C - \frac{m}{c}\sin\beta, \qquad D_t y = C' + \frac{m}{c}\cos i\cos\beta,$$
$$D_t z = C'' + \frac{m}{c}\sin i\cos\beta.$$

To find the values of the arbitrary constants, we observe that the velocities in the direction of the co-ordinate axes are each $= 0$, when the co-ordinates are each a maximum; hence,

$$D_\beta x = D_\beta r\cos\beta - r\sin\beta = 0, \qquad \text{or} \qquad \frac{D_\beta r}{r} = \tan\beta.$$

But $$\log r = \log a(1-e^2) - \log(1 + e\cos(\beta+\omega'));$$

and hence $$\frac{D_\beta r}{r} = \frac{e\sin(\beta+\omega')}{1+e\cos(\beta+\omega')} = \frac{\sin\beta}{\cos\beta}.$$

Reducing,

$$\sin\beta = e[\sin(\beta+\omega')\cos\beta - \cos(\beta+\omega')\sin\beta] = e\sin\omega'.$$

This value of $\sin\beta$ in the first of (87) gives,

$$0 = C - \frac{m}{c}e\sin\omega', \qquad \text{and} \qquad C = \frac{m}{c}e\sin\omega'.$$

In a similar manner we can find the value of C' and C'', and we shall thus obtain

(88) $D_t x = \frac{m}{c}(\sin\beta + e\sin\omega'),\quad D_t y = \frac{m}{c}\cos i(\cos\beta + e\cos\omega'),$

$$D_t z = \frac{m}{c}\sin i(\cos\beta + e\cos\omega').$$

If we put $D_t x = x'$, $D_t y = y'$, $D_t z = z'$, and take the sum of the squares of (88) we find, by recollecting that $\cos(\beta + \omega') = \cos\varphi$,

(89) $D_t s^2 = v^2 = s'^2 = x'^2 + y'^2 + z'^2 = \frac{m^2}{c^2}[1 + e^2 + 2e\cos\varphi].$

Put V_p for the velocity at the perihelion, and V_a at the aphelion, then, when $\varphi = 0$ in (89), $V_p^2 = \frac{m^2}{c^2}(1+e)^2$, and

(90) $$V_p = \frac{m}{c}(1+e).$$

When $\varphi = 180$, $V_a^2 = \frac{m^2}{c^2}(1-e)^2$, and

(91) $$V_a = \frac{m}{c}(1-e).$$

Whence we obtain

(92) $$\frac{V_p}{V_a} = \frac{1+e}{1-e} = \frac{\tan^2\frac{1}{2}\varphi}{\tan^2\frac{1}{2}E},$$

by (74).

When the planet is at the mean distance, $\cos\varphi = -e$, and hence

(93) $$V^2 = \frac{m^2}{c^2}(1-e^2) = \frac{m}{a},$$

since $c^2 = ma(1-e^2)$ by (51). Substituting this value of V^2 in (55), making $r = a$, and we have $h = -\frac{m}{a}$. This value of $h = C$ in (42) gives

(94) $$\frac{m}{a} = \frac{2m}{r} - (x'^2 + y'^2 + z'^2) = \frac{2m}{r} - s'^2.$$

27. The planetary *elements* are dependent on the velocity with which the bodies move, or the space passed over in a given time, and this is dependent on the attractive force of the sun, which is itself a function of the distance. We may hence regard each and

every *element* as a function of the distance, velocity, and time; and if we let A represent any one of the planetary elements, we can have the equations,

$$(95) \qquad A = f(x, y, z, x', y', z', t), \qquad A = f(r, s', t).$$

28. All the preceding 95 equations pertain to the motion of a single planet around the sun. When this is the case, A in (95) is strictly a constant; but if we introduce a second planet, it exercises an influence that causes the value of A to vary; and the extent of this variation will depend on the amount of the disturbing force. If we imagine an ellipse to be described, passing through the position of the disturbed planet (which call P), at any instant, it is evident that by an expansion, or contraction of this ellipse, as the case may require, we can suppose it to successively represent or pass through every point that P will occupy in completing a revolution. In this case it will be seen, that, at any instant (t), the distance of P from the sun, and also its velocity, will be the same in the *variable ellipse*, as in the orbit actually described; but the force that pertains to the two orbits will be different, otherwise the orbit would still be an ellipse. *The difference of the two forces will be the disturbing force.* If A varies, it will vary with the time. If we let D' represent the differential coefficient when the function (95) pertains to an ellipse, and D the differential coefficient when it pertains to the orbit actually described, the differentiation of the first of (95) gives on both suppositions,

$$(96) \quad 0 = (D'_t A) + D'_x A D'_t x + D'_y A D'_t y + D'_z A D'_t z + D'_{x'} A D'_t x' + D'_{y'} A D'_t y' + D'_{z'} A D'_t z'.$$

$$(97) \quad D_t A = (D_t A) + D_x A D_t x + D_y A D_t y + D_z A D_t z + D_{x'} A D_t x' + D_{y'} A D_t y' + D_{z'} A D_t z'.$$

$(D'_t A)$ and $(D_t A)$ are partial differential coefficients.

According to the hypothesis, $(D_t' A) = (D_t A)$, $D_t' x = D_t x$, &c., $D_x' A = D_x A$, &c., $D_{x'} A = D_{x'} A$, &c.; but $D_t x' = D_t^2 x =$ the force in the ellipse along the axis of x is not the same as $D_t x' = D_t^2 x =$ the force in the real orbit along the axis of x. The difference of (96) and (97) then gives

$$(98) \quad D_t A = D_{x'} A (D_t^2 x - D_t'^2 x) + D_{y'} A (D_t^2 y - D_t'^2 y) + D_{z'} A (D_t^2 z - D_t'^2 z).$$

If we put

$$(99) \qquad Q X, \; Q Y, \; Q Z,$$

for the disturbing force along the axes, we shall evidently have

$$Q X = D_t^2 x - D_t'^2 y, \quad Q Y = D_t^2 y - D_t'^2 y, \quad Q Z = D_t^2 z - D_t'^2 z.$$

Hence,

$$(100) \qquad D_t A = Q X . D_{x'} A + Q Y . D_{y'} A + Q Z . D_{z'} A.$$

This is a general formula, since it takes into account the whole amount of the disturbing force.

If we assume

$$(101) \quad A = A_x + A_y + A_z, \; D_t A = D_t A_x + D_t A_y + D_t A_z,$$

such that

$$(102) \quad D_t A_x = Q X . D_{x'} A, \; D_t A_y = Q Y . D_{y'} A, \; D_t A_z = Q Z . D_{z'} A,$$

we can compute the effects of the disturbing force in the direction of the co-ordinate axes (which we may regard as fixed in space) separately, and then by (101) we can conclude the joint or total effects. This, in general, will simplify the question.

If we treat the second equation of (95) in the same manner as that of the first, we shall find

$$(103) \qquad D_t A = Q S . D_s' A,$$

in which $$Q S = D_t^2 s - D_t'^2 s.$$

$Q\,S$ acts along the tangent of the curve, and it is hence the tangential force; and (103) will serve to compute the effects of this force.

29. $Q\,X$, $Q\,Y$ and $Q\,Z$ can represent the sum of the disturbing forces of any number of bodies, or any one of them. If the action of all the planets had to be considered at one time, the problem of perturbation would be very difficult; but from a principle in Mechanics, "immediately flowing from the primary relations between forces and the motions which they produce, that when a number of very minute forces act at once on a system, their joint effect is the sum, or aggregate of their separate effects, at least within such limits that the original relations of the parts of the system shall not have been materially changed by their actions"; and hence it simplifies the question very considerably to consider the action of each disturbing body separately. The question is thus reduced to the famous *Problem of Three Bodies*, which at present can only be solved approximately.

30. In the Solar System, the *inclination* of the planetary orbits to each other is so small, that it introduces little error in supposing the disturbed and the disturbing body to both lie in the same plane, except when we wish to compute the effect on the nodes and inclination.

31. If we substitute mr^{-2} for R, and $-ma^{-1}$ for $C=h$, in equation (28), and integrate, we find

$$D_t r^2 = \frac{2m}{r} - \frac{m^2}{r^2} - \frac{m}{a} = ma\left[\frac{2}{ar} - \frac{\quad}{ma\,r^2} - \frac{1}{a^2}\right]. \quad \text{But } \frac{c^2}{ma} = 1 - e^2.$$

$$\therefore\ D_t r^2 = m\left[\frac{a^2e^2 - (a-r)^2}{a\,r^2}\right]. \tag{104}$$

If we let $k^2 = a(1-e^2)$, and multiply (104) by $\frac{k^2}{e^2}$, we find

$$\frac{k^2 D_t r^2}{e^2} = 1 - \frac{(k^2-r)^2}{e^2 r^2}. \tag{105}$$

By collecting equations (76) and (94) we have

$$c = xy' - yx', \tag{106}$$

$$\frac{m}{a} = \frac{2m}{r} - (x'^2 + y'^2) = \frac{2m}{r} - s'^2 = \frac{2m}{k^2}[1 + e\cos(\lambda + \pi)] - (x'^2 - y'^2),$$

in which $z' = 0$, in accordance with Art. 30; and $\lambda + \pi = \varphi$ are so chosen that π is the longitude of the perihelion. Since

$$\frac{k^2}{r} = 1 + e\cos(\lambda + \pi), \qquad \cos(\lambda + \pi) = \frac{k^2 - r}{er};$$

and hence

$$\sin(\lambda + \pi) = \sqrt{\left(1 - \frac{(k^2 - r)^2}{e^2 r^2}\right)} = \frac{k D_t r}{e}, \tag{107}$$

by (105).

The manner in which we found equation (100) shows that $D_{x'} A$, $D_{y'} A$, $D_{z'} A$, are partial differential coefficients. The first of equations (106) will then give

$$D_{x'} c = -y, \qquad D_{y'} c = x,$$

and these values in (100), regarding $QZ = 0$, give

$$D_t c = -QX \cdot y + QY \cdot x. \tag{108}$$

The second of (106) gives $m D_{x'} a = 2a^2 x'$, $m D_{y'} a = 2a^2 y'$. These values in (100) give

$$D_t a = \frac{2a^2}{m}[QX \cdot D_t x + QY \cdot D_t y]. \tag{109}$$

We also have, from (106),

$$D_{x'} e = \frac{k^2 x'}{m\cos(\lambda + \pi)}, \qquad D_{y'} e = \frac{k^2 y'}{m\cos(\lambda + \pi)}.$$

These values, in (100), give

$$D_t e = \frac{k^2}{m\cos(\lambda + \pi)}[QX \cdot D_t x + QY \cdot D_t y] = \frac{k^2 e r}{2a^2(k^2 - r)} \cdot D_t a. \tag{110}$$

For the variation of π, (106) gives

$$D_{x'}\pi = -\frac{k^2 x'}{m e \sin(\lambda + \pi)} \qquad D_{y'}\pi = -\frac{k^2 y'}{m e \sin(\lambda + \pi)}.$$

These values, substituted in (100), and reducing by means of (107), we find

$$(111)\quad D_t \pi = -\frac{k}{m D_t r}[Q X . D_t x + Q Y . D_t y] = -\frac{k}{2 a^2 D_t r} . D_t a.$$

From (59) we have

$$(112)\qquad m T^2 = 4\pi^2 a^3, \qquad D_t T = \frac{6\pi^2 a^2}{m T} . D_t a.$$

The mean angular velocity, or *mean motion*, is

$$n = \frac{2\pi}{T}, \qquad n^2 = \frac{4\pi^2}{T^2}, \qquad n^2 a^3 = \frac{4\pi^2 a^3}{T^2} = m,$$

by (59). Hence,

$$(113)\qquad n^2 = m a^{-3}, \quad \text{and} \quad D_t n = -\frac{3m}{2 n a^4} . D_t a.$$

If we put M' for the mean longitude at a given epoch, and then $nt + M'$ will give the mean longitude for any time t, after; and, since π is the longitude of the perihelion, $nt + M' - \pi$ will give the mean anomaly at any time; and hence, by (78),

$$(114)\qquad n t + M' - \pi = E - e \sin E.$$

If M' varies, it will be in consequence of the variation of π and e, since nt and E are variable quantities in elliptic motion. Hence,

$$D_t M' = D_t \pi - \sin E\, D_t e = D_t \pi - \sqrt{\left(1 - \left(\frac{a-r}{a}\right)^2\right)}\, D_t e.$$

By substituting for $D_t \pi$ and $D_t e$, their values as given by (110) and (111), we have

$$(115)\quad D_t M' = -\left[\frac{k}{2 a^2 D_t r} + \frac{k^2 e r}{2 a^2 (k^2 - r)} \sqrt{\left(1 - \left(\frac{a-r}{a}\right)^2\right)}\right] D_t a.$$

32. To investigate the effects of the tangential force, we shall use the equation

$$\frac{m}{a} = \frac{2m}{k^2}\left[1 + e\cos(\lambda + \pi)\right] - S'^2. \tag{116}$$

For a, we have $'D_s a = \frac{2a^2 s'}{m}$, and hence, by (103),

$$D_t a = \frac{2a^2}{m} \cdot Q S \cdot D_t s. \tag{117}$$

For e, we have $D_s e = \frac{k^2 s'}{m\cos(\lambda+\pi)} = \frac{k^2 e r s'}{m(k^2 - r)}$. Hence, by (103),

$$D_t e = \frac{k^2 e r}{m(k^2 - r)} \cdot Q S \cdot D_t s. \tag{118}$$

For π, we have $D_s \pi = -\frac{k^2 s'}{m e \sin(\lambda+\pi)} = -\frac{k^2 s'}{m D_t r}$. Hence,

$$D_t \pi = -\frac{k^2}{m D_t r} \cdot Q S \cdot D_t s. \tag{119}$$

$$D_t M' = -\frac{k^2}{m}\left[\frac{1}{D_t r} + \frac{e r}{k^2 - r}\sqrt{\left(1 - \left(\frac{a-r}{a}\right)^2\right)}\right] \cdot Q S \cdot D_t s. \tag{120}$$

The preceding equations expressing the differential coefficients of the elliptic elements with respect to the time, give the momentary variations of these elements, and without the aid of the integral calculus, they are capable of furnishing us accurate ideas respecting the action of the disturbing force.

NOTES AND QUERIES.

I. *Note on a Question in Interest.* — In the March number of the Monthly, Prof. HOYT cites the following from ROBINSON'S Algebra: — "A man borrowed $ 60; what sum shall he pay daily to cancel the debt, principal and interest, in 60 days; interest at 10 per cent for

12 months of 30 days each?" The answers given in different editions are $\$1\frac{5}{600}$, and $\$1\frac{6}{7200}$; and Prof. HOYT asks, "Is either of them *correct*, or is the true answer $\$1\frac{6}{7259}$?"

Before answering this question, let us consider what rule or principle shall guide us in treating the commercial relations described in the problem. It is required to pay principal and interest. *What* interest? There are in common use two modes of adjusting the interest account, when an obligation is cancelled by partial payments. The one is described by the United States rule; the other by the Vermont Rule. According to the former, the interest to be cancelled is that which shall accrue on the unpaid parts of the principal during the intervals between the successive payments; according to the latter, the interest to be paid is the excess of the interest of the given sum for the whole time, above that accruing on the various payments from the dates at which they were severally made to the time of settlement. These two methods give different results. The problem under consideration involves precisely those commercial relations which require the application of one or the other of these rules. ROBINSON has adopted, in his solutions, neither of these rules, and both of his answers are erroneous. Prof. HOYT has based his solution on the Vermont rule, and has given the correct answer according to that rule. But it seems to me, that in all examples of this nature, where this condition is not defined in the problem, the United States rule should be applied. Not only has it the higher commercial authority, but the analysis by this principle is more pertinent and rational. The given sum is made to draw interest for the first interval. Then a payment is made, which cancels the interest due and a part of the principal. There is now left a balance of the principal on which interest will accrue for another interval; and so on.

To obtain a general solution, let p represent the principal, r the

interest of $1 for one interval, y one of the equal payments, and n the number of payments. Then

(1) $$y = \frac{p\,r\,(1+r)^n}{(1+r)^n - 1},$$ by the United States rule,

(2) $$y = \frac{2\,p\,(1+r\,n)}{n(n-1)\,r+2n}$$ by the Vermont rule.

The answer to this problem by equation (1) is $ 1.0084984+.

It is but just to say that ROBINSON is not the only author in fault in respect to examples of this kind. In DAVIES's University Arithmetic we find the following: —

"A farmer bought a piece of land for $ 1,500, and agreed to pay principal and interest in five equal annual instalments; how much was the annual payment?"

A reference to the Key shows that 7 per cent interest is understood. The answer given is $ 360.56. Now this answer cannot be verified by any recognized rule for partial payments. The annual payment required to cancel an obligation of $ 1,500 in five years is $ 366.08, interest reckoned by the United States rule, and $ 342.10, interest reckoned by the Vermont rule. — Prof. J. C. PORTER, Liberal Institute, Clinton, New York.

II. *Note on the General Theory of Equations.* — In developing the general theory of equations, it is usual to multiply together the transposed equations formed by x with each of its assigned values, and to treat x in the product, though of different values, as one quantity. The student is often perplexed by this process. Why set down as x^2 the product of x having one value, by x having another value? Why unite the several terms containing the first power of x in the product, when in fact x has different values in those terms? And why will each one of the assigned values of x

verify the resulting equation by being substituted throughout, when the terms which are written as powers of x are the products of factors originally different, or the sums of such products?

In order to explain this, let us at first keep all the monomial factors distinct. Multiplying together $x+a=0$, $x'+b=0$, $x''+c=0$, &c., we obtain the successive equations,

$$(x+a)(x'+b)=0.$$
$$(x+a)(x'+b)(x''+c)=0.$$
$$(x+a)(x'+b)(x''+c)(x'''+d)=0, \text{ \&c.}$$

Now it is evident that whatever be the number of binomial factors used as above, we may substitute x for x', x'', &c., or any one of these for all the others, without destroying the equation; for in each case one of the factors, and, consequently, the whole product, will still remain equal to zero.

For an example, take the second of the above equations. Expanding this, we obtain an equation having eight terms. But since $x+a=0$, we have $(x+a)(x+b)(x+c)=0$; and since $x'+b=0$, $(x'+a)(x'+b)(x'+c)=0$; and since $x''+c=0$, $(x''+a)(x''+b)(x''+c)=0$. Expanding these and uniting like terms, we get from one equation three numerically distinct equations, of four terms each.

$$x^3+Ax^2+Bx+C=0.$$
$$x'^3+Ax'^2+Bx'+C=0.$$
$$x''^3+Ax''^2+Bx''+C=0.$$

All these are obviously contained in the following general formula, where X is to be understood as having three values:—

$$X^3+AX^2+BX+C=0.$$

Thus it appears that the equation obtained by blending x, x', x'',

&c. in the product, though different from that which we should obtain by keeping them distinct, is nevertheless a true equation, and one which each of the originally assigned values of x will verify. — Prof. E. W. Evans, Marietta College, Ohio.

III. *Trigonometrical Formulas.* — The values of sin $(A \pm B)$, cos $(A \pm B)$ in terms of the sines and cosines of A and B may be deduced as follows: Take two triangles whose sides are a, b, c, and opposite angles A, B, C, and suppose the angle A in one of them to be obtuse. From the vertical angle C, drop a perpendicular, d, on the base in one case, and on the base produced in the other. The triangles stand on the base AB, and D is the foot of the perpendicular. If $A = CAD$ and $B = CBD$ in both cases, then sin $C = \sin(A+B)$ in the first, and sin $C = \sin(A-B)$ in the second, and

$$\sin C = \sin(A \pm B) = \frac{dc}{ab} = \frac{d(DB \pm AD)}{ab} = \frac{d}{b} \cdot \frac{DB}{a} \pm \frac{d}{a} \cdot \frac{AD}{b}$$
$$= \sin A \cos B \pm \sin B \cos A,$$

since, in both cases,

$$\sin A = \frac{d}{b}, \quad \sin B = \frac{d}{a}, \quad \cos A = \frac{AD}{b}, \quad \cos B = \frac{DB}{a}.$$

In the first case, $\cos C = -\cos(A+B)$, and, in the second, $\cos C = \cos(A-B)$. But $\cos C = \frac{a^2+b^2-c^2}{2ab}$, and therefore

$$\cos C = \cos(A \pm B) = \mp \frac{a^2+b^2-c^2}{2ab} = \frac{AD}{b} \cdot \frac{BD}{a} \mp \frac{d}{b} \cdot \frac{d}{a}$$
$$= \cos A \cos B \mp \sin A \sin B,$$

since

$$b^2 = d^2 + AD^2, \quad a^2 = d^2 + BD^2, \quad c^2 = (BD \pm AD)^2.$$

— Prof. R. M. Moore, M'Kendree College, Lebanon, Illinois.

Mathematical Monthly Notices.

The Claim of LEIBNITZ *to the Invention of the Differential Calculus, by* DR. H. SLOMAN. *Translated from the German with considerable Alterations and new Addenda by the Author.* Cambridge: MACMILLAN & Co., and 23 Henrietta Street, Covent Garden, London. 1860. 4to. pp. 157.

A few years ago PROFESSOR DE MORGAN remarked, that "all the notions as to what is fair and what is unfair in controversy have undergone much change since the commencement of the last century." This remark may be true as to the world in general, but it is evidently false as to DR. SLOMAN. Into the great controversy concerning the invention of the Differential Calculus, which of late years has had so much new interest thrown around it, our author enters with a bitterness that KEILL and FATIO DE DUILLIER might have envied. He espouses the Newtonian side of the contest, and endeavors to turn back the tide which has of late been so steadily setting towards a complete vindication of the fame of LEIBNITZ. The researches of GERHARDT, DE MORGAN, and others, and the new edition of the Commercium Epistolicum by BIOT and LEFORT, were fast establishing the claim of LEIBNITZ to an independent invention of the Differential Calculus, as well as to the sole invention of the notation that gave to it its vast power. But while every true man was rejoicing to see this vindication of a great intellect, our author enters the field with very different feelings. With very little in the way of new facts, and that little unimportant, his book is filled with misrepresentations, garbled extracts, mean insinuations, and base perversions. The only thing it exhibits fairly, it exhibits undesignedly, namely, the moral character of its author, showing it to be of that low type that labors to put the worst construction on every word and act of an opponent.

We do not purpose to enter into an examination of this book in detail. To do so would involve a review of the whole controversy, and would, moreover, ascribe to the work an importance which it does not deserve. We shall, however, examine a preliminary page, which purports to give extracts from GERHARDT, EDLESTON, and LEIBNITZ, and which stands thus: —

GERHARDT: "So it is Leibnitz — — to whom "we owe this. The Chapter in which the question "respecting the first discoverer of the higher An- "alysis has till now been agitated, now disappears "from the history of the mathematical sciences. "The battle of more than one hundred years, "about the first discoverer of the Differential Cal- "culus, is now at its end." *Gerhardt*, II. page 62.

GERHARDT: "So ist es Leibnitz — — dem wir "dies verdanken. Das Capitel, in welchem die "Frage über den ersten Entdecker der höheren "Analysis bisher erörtert wurde, verschwindet for- "tan aus der Geschichte der mathematischen Wis- "senschaften. Der mehr als hundert jährige Streit "über den ersten Entdecker der Differenzialrech- "nung ist zu Ende." *Gerh. Abh.* II. S. 62.

EDLESTON: Synoptical view of Newton's life:

1666 Octob. Small tract on fluxions and fluents with their applications to a variety of problems on tangents, curvature, areas, lengths, and centres of gravity of curves.

1666 November. Small tract similar to the preceding but apparently more comprehensive. (Notation by points in first and second fluxions, Basis of his larger tract of 1671.) *Edleston* in his Correspondence of Sir I. Newton, 1850, page 21.

LEIBNITZ: Cum Parisios apulissem anno Christi 1672 eram ergo — — in pene dixerim superba Mathescos ignorantia. *Leibnitz* in Gerhardt's Pamphlet, I. page 29 et 30, line 2.

The reader would suppose from this, that GERHARDT here claims that LEIBNITZ was "the first discoverer of the Differential Calculus," which claim is disposed of by showing, in the succeeding extracts, that NEWTON had his method of Fluxions in 1666, six years before the time

when LEIBNITZ declared his ignorance of Mathematics. We pass over the mistakes in the last extract, of *apulissem* for *appulissem*, and *ergo* for *ego*, and also the bringing together into one sentence of the parts of widely separated sentences, because it happens that here the sense is not essentially changed. But it is otherwise with the first extract. The first words, "So ist es LEIBNITZ," are not in GERHARDT; the next four, "dem wir dies verdanken," are altered from a sentence on the preceding page, and the remainder of the extract is part of a sentence separated from the rest by more than half a page. The meaning of GERHARDT is, by this ingenious process, entirely perverted. The passage occurs in a chapter headed, "*The Discovery of the Algorithm of the Higher Analysis by* LEIBNITZ," and has reference to the system of notation invented by him. No one could mistake this; for GERHARDT'S words in the first sentence quoted from are, "the *notation* which we owe to Leibnitz" (*die Bezeichnungsweise die wir Leibnitz verdanken*). GERHARDT does not deny, and no one, we presume, denies, that NEWTON had made the discovery of what was afterwards called his Method of Fluxions (but without any general notation) several years before LEIBNITZ began the study of the higher mathematics; and his object in this place is simply to claim for LEIBNITZ the invention of his system of notation. After showing from his manuscripts the successive steps in the development of the subject, as they presented themselves to the mind of LEIBNITZ, GERHARDT stops a moment to consider the importance of the invention, and closes with the opinion, that every suspicion of plagiarism on the part of LEIBNITZ must now vanish, and, therefore, that the long controversy may now be considered at an end. We give below a translation of the whole passage, — interesting in itself, — that our readers may judge for themselves how fairly the extracts of Dr. SLOMAN are made.

"This is the origin of that notation to which the Higher Analysis owes its growth and wonderful perfection, — and the twenty-ninth of October, 1675, is the memorable day that saw its birth. The importance of the subject demands that we should pause an instant, in order to rightly estimate this significant moment. The bold speculations of KEPLER and the *Method of Indivisibles* of CAVALIERI had furnished the means of treating more easily than before the problems of higher geometry. In addition, WALLIS, in his *Arithmetica Infinitorum*, had shown how the general methods of representing magnitudes, introduced by VIETA and DESCARTES, could be used in quadratures and cubatures. By these means a great many particular results were reached that are now obtained by the Higher Analysis; but there was wanting a common bond to free them from their isolation, and unite them under a single point of view. This the natural and most happy notation which we owe to LEIBNITZ effected at a single stroke. It has been universally acknowledged by the great mathematicians of succeeding times, that the Algorithm of LEIBNITZ called forth the surprisingly rapid progress that was made in enlarging and applying the Higher Analysis, and brought about the completion of that truly noble edifice, to survey which and to explore whose various parts are given only to him in whom eminent ability is found united with the highest power of attention and unwearied perseverance. It is not to be overlooked, that LEIBNITZ, led by his lucky genius, came almost accidentally upon the introduction of the algorithm of the Higher Analysis; nor, on the other hand, can it be denied, that being prepared for it by his earlier studies, and surveying the field of science, he recognized at once, though not in its full extent, the importance of the happy discovery, and knew how to employ it in the most skilful manner. This, however, every one must openly concede, — that LEIBNITZ, in introducing his new algorithm, proceeded independently throughout, without even the slightest external influence; an unprejudiced examination of his manuscripts forces us to this conclusion. All doubt, every suspicion of plagiarism, is removed by the manuscripts which are here for the first time connectedly published, and the chapter in which the question respecting the first discoverer of the Higher Analysis has been till now agitated, henceforth disappears from the history of the mathematical sciences. The battle of more than one hundred years about the first discoverer of the Differential Calculus is now at an end."

Let us hope that the spirit which animates the work of DR. SLOMAN will have few imitators. Let us rather cherish and defend the fame of both NEWTON and LEIBNITZ, recollecting that the honor to our race is doubled, if two, instead of one, are shown to be capable of a great invention. Let us be ready to rejoice whenever the impartial hand of later research reveals a new brightness in those great names which the envy and malice of contemporaries sometimes obscure.

H.

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PRIZE PROBLEMS FOR STUDENTS.

I. If a quadrilateral circumscribe a circle, its diagonals, and the lines joining the points of contact of opposite sides, meet in a point.

II. Two men, A and B, agree to perform two equal pieces of work. A begins one piece, and B the other; and after working p days they exchange pieces. A finishes his piece in q days, and then returns and helps B; and after working together r days they finish his piece. How long would each be in doing either piece? If A is twice as good a workman as B, show that $q = p + 3r$.

III. Prove that, in any triangle,

(1) $\cot B - \cot A = \dfrac{a^2 - b^2}{a\,b\,\sin c}$, (2) $\dfrac{\sin(\frac{1}{2}A + B)}{\sin \frac{1}{2}A} = \dfrac{b+c}{a}$,

(3) $\text{Area} = \sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C \left(\dfrac{a^2}{\sin A} + \dfrac{b^2}{\sin B} + \dfrac{c^2}{\sin C}\right)$.

IV. Find the number of conditions that must be satisfied by the coefficients of the general equation of the nth degree, that it may represent a series of parallel lines at equal distances from each other.

V. The population of a town at the end of any year can be found by subtracting eleven times the population at the end of the previous year from ten times the population at the end of the succeeding year. Nine years ago the population was 1210, and eleven years 1000. Show that it increases in geometrical progression.

Solutions of these problems must be received by August 1, 1861.

TO CHANGE A SERIES INTO A CONTINUED FRACTION.

By A. Hall, Cambridge, Mass.

Gauss has shown, in his "Disquisitiones Generales circa seriem infinitam, etc.," that nearly all the series which commonly occur in algebraic analysis are contained as special cases in the following one: —

$$(1)\quad 1+\frac{\alpha\beta}{1.\gamma}x+\frac{\alpha(\alpha+1)\beta(\beta+1)}{1.2.\gamma(\gamma+1)}x^2+\frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1.2.3.\gamma(\gamma+1)(\gamma+2)}x^3+\text{etc.}$$

The quantities α, β, γ, x are called the *elements* of this series, and are distinguished in their order as the first, second, third, and fourth elements.

The following are some of the principal properties of this series.

Since the first and second elements enter the series symmetrically, we can change places with them, and the value of the series will not be altered. Denoting, therefore, the series by $F(\alpha, \beta, \gamma, x)$, we shall have $F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$.

By giving to the elements α, β, γ definite values, the series becomes a function of a single variable x, and is evidently broken off after the $1-\alpha$th or the $1-\beta$th term, if $\alpha-1$ or $\beta-1$ is a negative whole number; but in all other cases the series extends to infinity. In the first case, the series becomes a rational algebraic function; but in the last, it is commonly a transcendental function. The third element, γ, can be neither a negative whole number nor $= 0$, lest we fall upon terms infinitely great.

The coefficients of the powers x^m, x^{m+1} are as $1+\frac{r+1}{m}+\frac{r}{m^2}$ is to $1+\frac{\alpha+\beta}{m}+\frac{\alpha\beta}{m^2}$, and therefore approach a ratio of equality as m is increased. Hence the convergency of the series depends on the value given to the fourth element, x, and the series will always converge if x has a real value, positive or negative, and less than unity.

By giving to the elements proper values, it is easy to represent some of the more common series. Thus,

$$(a) \qquad (t+u)^n = t^n . F\left(-n, \beta, \beta, -\frac{u}{t}\right),$$

when the element β is still arbitrary.

$$(b) \qquad \log(1+t) = t . F(1, 1, 2, -t),$$

$$(c) \qquad \log \frac{1+t}{1-t} = 2t . F(\tfrac{1}{2}, 1, \tfrac{3}{2}, t^2),$$

$$(d)\ e^t = F\left(1,k,1,\frac{t}{k}\right) = 1 + t . F\left(1,k,2,\frac{t}{k}\right) = 1 + t + \tfrac{1}{2}t^2 . F\left(1,k,3,\frac{t}{k}\right), \text{etc.},$$

when e denotes the base of hyperbolic logarithms and k is a number infinitely great. Also,

$$(e) \quad \text{arc} . t = \sin t . F(\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{3}{2}, \sin^2 t) = \sin t \cos t . F(1, 1, \tfrac{3}{2}, \sin^2 t).$$

The series (1) possesses the important property, that, if we subtract it from a similar series, the remainder is a series of the same form. Writing, in (1), $\beta + 1$ and $\gamma + 1$ for β and γ, we shall have

$$F(\alpha, \beta+1, \gamma+1, x) = 1 + \frac{\alpha(\beta+1)}{1.(\gamma+1)} x + \frac{\alpha(\alpha+1)(\beta+1)(\beta+2)}{1.2.(\gamma+1)(\gamma+2)} x^2 + \text{etc.},$$

and, subtracting (1),

$$F(\alpha,\beta+1,\gamma+1,x) - F(\alpha,\beta,\gamma,x) = \frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)} . \left[1 + \frac{(\alpha+1)(\beta+1)}{1.(\gamma+2)} x + \frac{(\alpha+1)(\alpha+2(\beta+1)(\beta+2)}{1.2.(\gamma+2)(\gamma+3)} x^2 + \text{etc.},\right.$$

or,

$$(2)\ F(\alpha,\beta+1,\gamma+1,x) - F(\alpha,\beta,\gamma,x) = \frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)} . F(\alpha-1,\beta-1,\gamma-2,x).$$

And in the same way

$$(3)\ F(\alpha-1,\beta+1,\gamma,x) - F(\alpha,\beta,\gamma,x) = \frac{(\alpha-\beta-1)x}{\gamma} . F(\alpha,\beta+1,\gamma+1,x).$$

Two more similar equations can be immediately obtained by interchanging α and β; and, by the same process by which equations (2)

and (3) have been obtained, we may find many other such equations. This last property can be used to transform the series (1) into a continued fraction.

Designating $\frac{F(\alpha, \beta+1, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)}$ by $G(\alpha, \beta, \gamma, x)$, we have, by interchanging α and β,

$$\frac{F(\beta, \alpha+1, \gamma+1, x)}{F(\beta, \alpha, \gamma, x)} = G(\beta, \alpha, \gamma, x);$$

and since $F(\alpha, \beta, \gamma, x) = F(\beta, \alpha, \gamma, x)$, we have

$$F(\beta, \alpha+1, \gamma, x) = F(\alpha+1, \beta, \gamma+1, x),$$

and therefore

$$\frac{F(\alpha+1, \beta, \gamma+1, x)}{F(\alpha, \beta, \gamma, x)} = G(\beta, \alpha, \gamma, x). \tag{4}$$

Dividing equation (2) by $F(\alpha, \beta+1, \gamma+1, x)$, we have, by equation (4),

$$1 - \frac{1}{G(\alpha, \beta, \gamma, x)} = \frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)} \cdot G(\beta+1, \alpha, \gamma+1, x),$$

where, in $G(\beta+1, \alpha, \gamma+1, x)$, $\beta+1$ and $\gamma+1$ are written for β and γ. Hence,

$$G(\alpha, \beta, \gamma, x) = \frac{1}{1 - \frac{\alpha(\gamma-\beta)x}{\gamma(\gamma+1)} \cdot G(\beta+1, \alpha, \gamma+1, x)}.$$

Again, by interchanging α and β, and writing $\beta+1$ and $\gamma+1$ for β and γ, we get

$$G(\beta+1, \alpha, \gamma+2, x) = \frac{1}{1 - \frac{(\beta+1)(\gamma+1-\alpha)x}{(\gamma+1)(\gamma+2)} \cdot G(\alpha+1, \beta+1, \gamma+2, x)},$$

$$G(\alpha+1, \beta+1, \gamma+2, x) = \frac{1}{1 - \frac{(\alpha+1)(\gamma+1-\beta)x}{(\gamma+2)(\gamma+3)} \cdot G(\beta+2, \alpha+1, \gamma+3, x)},$$

etc.

Whence, by repeated substitutions of the function G, there will result for $G(\alpha, \beta, \gamma, x)$ the continued fraction,

$$(5)\qquad \frac{F(\alpha,\beta+1\,\gamma+1,x)}{F(\alpha,\beta,\gamma,x)} = \cfrac{1}{1-\cfrac{ax}{1-\cfrac{bx}{1-\cfrac{cx}{1-\cfrac{dx}{1-\text{etc.}}}}}},$$

where

$$a = \frac{\alpha(\gamma-\beta)}{\gamma(\gamma+1)}, \qquad b = \frac{(\beta+1)(\gamma+1-\alpha)}{(\gamma+1)(\gamma+2)},$$

$$c = \frac{(\alpha+1)(\gamma+1-\beta)}{(\gamma+2)(\gamma+3)}, \qquad d = \frac{(\beta+2)(\gamma+2-\alpha)}{(\gamma+3)(\gamma+4)},$$

$$e = \frac{(\alpha+2)(\gamma+2-\beta)}{(\gamma+4)(\gamma+5)}, \qquad f = \frac{(\beta+3)(\gamma+3-\alpha)}{(\gamma+5)(\gamma+6)},$$

etc., etc.,

in which the law of progression is obvious. It is evident that this continued fraction will stop if either of the numbers $\alpha, \beta, \gamma-\alpha, \gamma-\beta$ is a negative whole number, but otherwise it will extend to infinity.

If in formula (5) we put $\beta = 0$, whence $F(\alpha,\beta,\gamma,x) = 1$, and write $\gamma - 1$ for γ, we shall have

$$(6)\quad F(\alpha,1,\gamma,x) = 1 + \frac{\alpha}{\gamma}x + \frac{\alpha(\alpha+1)}{\gamma(\gamma+1)}x^2 + \frac{\alpha(\alpha+1)(\alpha+2)}{\gamma(\gamma+1)(\gamma+2)}x^3 + \text{etc.}$$

$$= \cfrac{1}{1-\cfrac{ax}{1-\cfrac{bx}{1-\cfrac{cx}{1-\cfrac{dx}{1-\text{etc.}}}}}},$$

where we now have

$$a = \frac{\alpha}{\gamma}, \qquad b = \frac{\gamma-\alpha}{\gamma(\gamma+1)},$$

$$c = \frac{(\alpha+1)\gamma}{(\gamma+1)(\gamma+2)}, \qquad d = \frac{2(\gamma+1-\alpha)}{(\gamma+2)(\gamma+3)},$$

$$e = \frac{(\alpha+2)(\gamma+1)}{(\gamma+3)(\gamma+4)}, \qquad f = \frac{3(\gamma+2-\alpha)}{(\gamma+4)(\gamma+5)},$$

etc.

By means of equation (6) we can express the equations (*a*), (*b*), (*c*), etc., in continued fractions. Putting $t = 1$, $\beta = 1$, we have, for the binomial,

$$(1+u)^n = \cfrac{1}{1-\cfrac{nu}{1+\cfrac{\frac{n+1}{2}u}{1-\cfrac{\frac{n-1}{2.3}u}{1+\cfrac{\frac{2(n+2)}{3.4}u}{1-\cfrac{\frac{2(n-2)}{4.5}u}{\text{etc.}}}}}}}$$

Equation (*c*) gives

$$\log.\frac{1+t}{1-t} = \cfrac{2t}{1-\cfrac{\frac{1}{3}t^2}{1-\cfrac{\frac{2.2}{3.5}t^2}{1-\cfrac{\frac{3.3}{5.7}t^2}{1-\cfrac{\frac{4.4}{7.9}t^2}{\text{etc.}}}}}}$$

By changing in this last equation the signs — into +, we shall have a continued fraction for arc . tan *t*. Again, if in the preceding fraction for the binomial $(1+u)^n$ we put $n = k$, and $u = \frac{t}{k}$, we shall have, by making k infinite, a continued fraction for the exponential e^t.

If we put $\alpha = 3$, $\gamma = \frac{5}{2}$, in equation (6), we have immediately the first continued fraction in article 90, *Theoria Motus*. For the second one of this article, let

$$Q = 1 - \cfrac{\frac{5.8}{7.9}x}{1-\cfrac{\frac{1.4}{9.11}x}{\text{etc.,}}}$$

and we have

$$x - \xi = \frac{x}{1 + \frac{2x}{35Q}} = \frac{xQ}{Q + \frac{2x}{35}},$$

and therefore

$$\xi = \frac{\frac{2}{35}x^2}{Q + \frac{2}{35}x},$$

which is the second formula, or [13]. The third is found in the following manner.

Putting
$$R = 1 - \cfrac{\frac{1.4}{7.9}x}{1 - \cfrac{\frac{5.8}{9.11}x}{1 - \cfrac{\frac{3.6}{11.13}}{1 - \frac{7.10}{13.15}x}}}, \text{ etc.,}$$

we find, by equation (5),

$$\frac{1}{R} = G(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x), \qquad \text{and} \qquad \frac{1}{Q} = G(\tfrac{5}{2}, -\tfrac{1}{2}, \tfrac{7}{2}, x).$$

Hence,
$$R \,.\, F(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x) = F(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x),$$
$$Q \,.\, F(\tfrac{5}{2}, \tfrac{1}{2}, \tfrac{9}{2}, x) = F(\tfrac{5}{2}, -\tfrac{1}{2}, \tfrac{7}{2}, x);$$

or, by interchanging the first and second elements in this last equation,

$$Q \,.\, F(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x) = F(-\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, x).$$

But by equation (3) we have

$$F(-\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{7}{2}, x) - F(\tfrac{1}{2}, \tfrac{3}{2}, \tfrac{7}{2}, x) = -\tfrac{4}{7}x \,.\, F(\tfrac{1}{2}, \tfrac{5}{2}, \tfrac{9}{2}, x),$$

and therefore, substituting and dividing by $F(\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, x)$, we have

$$Q = R - \tfrac{4}{7}x;$$

and substituting this in the preceding value for ξ,

$$\xi = \frac{\frac{2}{3}\frac{2}{5}x^2}{R - \frac{1}{3}\frac{1}{5}x},$$

which is the last formula.

The preceding method of transforming a series into a continued fraction was first given by GAUSS, in the Memoir referred to at the beginning of this article, and which is printed in the Memoirs of the Göttingen Society, 1812. See also, on the subject, CRELLE'S Journal, Vol. X., where there is a very elaborate article by STERN, GERGONNE'S *Annales*, Vol. IX., and a little book called *Algebraische Analysis*, by Dr. O. SCHLÖMILCH.

ON THE GEOMETRICAL CONSTRUCTION OF CERTAIN CURVES BY POINTS.

By H. A. NEWTON, Professor of Mathematics in Yale College.

[Continued from page 244.]

33. THE method given by CHASLES for constructing the cubic through nine points is such that with the straight edge alone can be constructed the point where a straight line through two points again meets the curve.* Hence any number of points in the problem of the last article can be constructed by the straight edge alone.

34. The construction of a biquadratic to pass through fourteen single points is given by DE JONQUIERES (LIOUVILLE, 2d series, Vol. I. p. 416). Hence any construction of a curve which can be reduced to the construction of such a biquadratic is possible.

35. To construct a curve of the nth order (n being not less than 5 and not greater than 15), having given $(15 - n)$ single points,

* This fact is not mentioned by CHASLES, though he must have known it.

3 double points, $(n-5)$ triple points, and 1 point of the $(n-4)$th order.

Take the point of the $(n-4)$th order, a triple point, and a double point as base points, and the construction depends on that of a curve of the $(n-1)$st order, $(15-(n-1))$ single points, three double points, $(n-6)$ triple points, and one point of the $(n-5)$th order; that is, the construction depends on that of the curve of the $(n-1)$st order, to satisfy conditions like those of the original problem. By repeating the process, the construction depends, finally, on that of a curve of the fifth degree, having three given double points and eleven single points. Taking the three double points as base points, the construction depends on that of the bi-quadratic through fourteen points.

36. These theorems can be multipled indefinitely. The following table is arranged to show in what cases curves of the fourth, fifth, sixth, seventh, and eighth degrees can be constructed. The asterisks indicate the constructions that are possible with the straight edge alone.

	I.	II.	III.	IV.	
4	14				
	11	1			
	8	2			*
	5	3			*
	8		1		*
5	11	3			
	8	4			
	5	5			*
	2	6			*
	11	1	1		
	8	2	1		*
	5	3	1		*
	10			1	*
6	9	4	1		
	6	5	1		
	3	6	1		*
		7	1		*
	12	1	2		
	9	2	2		
	6	3	2		*
	3	4	2		*
	9		3		*

	I.	II.	III.	IV.	V.	
6	6	1	3			*
	11	2		1		
	8	3		1		*
	5	4		1		*
	12				1	*
7	8	3	3			
	5	4	3			
	2	5	3			*
	8	1	4			
	5	2	4			*
	2	3	4			*
	7	6		1		
	4	7		1		
	1	8		1		*
	10	3	1	1		
	7	4	1	1		
	4	5	1	1		*
	1	6	1	1		*
	13		2	1		
	10	1	2	1		
	7	2	2	1		*
	4	3	2	1		*

	I.	II.	III.	IV.	V.	VI.	
7	7		3	1			*
	11	3			1		
	8	4			1		*
	5	5			1		*
	14					1	*
8	8		6				
	5	1	6				
	2	2	6				*
	2		7				*
	7	3	3	1			
	4	4	3	1			
	1	5	3	1			*
	10		4	1			
	7	1	4	1			
	4	2	4	1			*
	1	3	4	1			*
	4		5	1			*
	9	3	1	2			
	6	4	1	2			
	3	5	1	2			*
		6	1	2			*
	9	1	2	2			

	I.	II.	III.	IV.	V.	VI.	VII.	
8	6	2	2	2				*
	3	3	2	2				*
	14			3				
	11	1		3				
	8	2		3				*
	5	3		3				*
	8		1	3				*
	5	8			1			
	2	9			1			
	8	5	1		1			
	5	6	1		1			
	2	7	1		1			*
	11	2	2		1			
	8	3	2		1			
	5	4	2		1			*
	2	5	2		1			*
	11		3		1			
	8	1	3		1			*
	5	2	3		1			*
	11	4				1		
	8	5				1		*
	5	6				1		*
	16						1	*

The arrangement of the table will be evident from an example. Thus, the last line of the third column indicates that a curve of the eighth degree can be constructed, having given nine single points, one double point, two triple points, and two quadruple points.

37. Curves, *properly*, of the degree mentioned in the first column of the table below, and having the number of multiple points indicated, are impossible.

	II.	III.	IV.	V.	VI.
6	5	2			
	5		1		
7	4	2	1		
	6			1	
8	1	5	1		
	4	2	2		
	3	3		1	
	7				1

For in each case, if the largest multiple points be taken as base points, and the largest multiple points of the conjugate curve be again taken as base points, and so on, we have at last a conic with a double point, or a cubic with two double points, either of which is impossible.

38. The given points have been thus far supposed all real. But they may also be imaginary in couples.

Two imaginary points may be considered as given when they are the points of intersection of a given straight line with a conic section, of which five points are given. For these data enable us to determine with the rule and compasses the various real relations of the imaginary points.

The method of determining two imaginary points on a given line by their middle point, and the product of their distances to a given point of the line, is not suitable to constructions in which the straight edge alone is to be used. With the compasses, however, we can describe a circle that shall pass through the points thus given. Five points on this circle can be used in these constructions.

39. Problem. — To construct a sixth point of a conic that shall pass through two imaginary and three real points.

Let the two imaginary points be the intersection of the straight line $\mathfrak{L}$ with the conic $a\,b\,c\,d\,e$, and let the three other given points

be P, Q, and R. On $\mathfrak{L}$ take three points, m, n, and o, and construct (Art. 25) the points m', n', and o', where the three conics, $abcdm$, $abcdn$, and $abcdo$, meet again the line $\mathfrak{L}$. Draw the straight line ae, and construct on this line the three points m'', n'', and o'', where the same three conics cut it.

Through P draw any straight line, and on this line construct the points m''', n''', and o''', where the three conics $PQRmm'$, $PQRnn'$, $PQRoo'$ cut it. Then on this line construct a fourth point, e', such that the anharmonic function of m''', n''', o''', e' is equal to that of m'', n'', o'', e. (*Géom. Sup.*, 113.) The point e' is a required point.

For the series of conics circumscribing the quadrilateral $abcd$ cuts $\mathfrak{L}$ in a series of couples of points in involution, of which the two given imaginary points form one couple, and the middle points of these couples correspond anharmonically with the series of points where the same conics cut the line ae. Moreover, the conics through PQR, and the couples of points in involution on $\mathfrak{L}$, pass all through a fourth point, S, and the middle points of the segments on $\mathfrak{L}$ correspond anharmonically to the points where these conics cut the line $Pm'''n'''o'''e'$. Therefore the two conics $PQRSe'$ and $abcde$ cut $\mathfrak{L}$ in the same two points, that is, in the given imaginary points.

The construction of the point e' is evidently possible with the straight edge alone.

40. Problem. — To construct a point on a conic that shall pass through a given real point and two given couples of imaginary points.

If the points Q and R are imaginary, the methods of the preceding article enable us to determine the points where the conics $PQRmm'$, $PQRnn'$, and $PQRoo'$ cut a line through S, since three given points of each conic are real. Hence the point e' can be constructed.

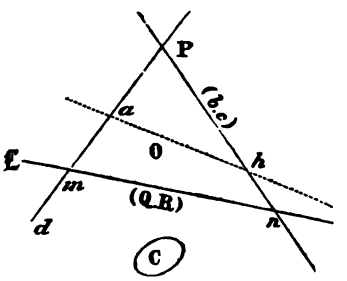
41. Hence the points conjugate to two given imaginary points may be constructed.

Construct two points, e' and e'', on the conic through $P\,Q\,R$, and the two imaginary points (39, 40). Construct, then, the conic conjugate to $\mathfrak{L}$, and the straight line conjugate to the conic $P\,Q\,R\,e'\,e''$. The intersections of this line with the conic will evidently be the points required.

They are evidently imaginary, for, if they were real, their conjugates, that is, the given points, would be real.

42. Two of the base points may also be imaginary, and yet the preceding constructions be possible. The conjugate to a real point will still be real.

Let Q and R be the intersections of the straight line $\mathfrak{L}$ (Fig. 5) with the conic C. Having assumed a, draw $P\,m$ cutting $\mathfrak{L}$ in m and find d, so that the segments $a\,d$ and $P\,m$ cut each other harmonically. The points a and d will be two real angles of the quadrilateral $a\,b\,c\,d$ (Art. 3), and the other angles, b and c, will be the imaginary points of intersection of the lines from a and d through Q and R.



If we draw the polar of the point m with respect to the conic C, and through n, where it cuts $\mathfrak{L}$, draw $P\,n$; then will $P\,n$ be the straight line through the imaginary points b and c. For the segments $m\,n$ and $Q\,R$ cut each other harmonically, according to the theory of poles and polars, and hence $P\,n$ is the polar of the point m with respect to the angle $Q\,P\,R$, and therefore passes through b and c (Art. 3).

43. To find the conjugate to a point, O, on the line $P\,d$, find the point O' such that $O\,O'$ cuts $a\,d$ harmonically (Art. 7).

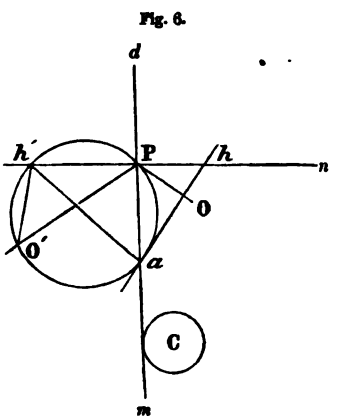
44. To find a conjugate to any point, O, on $P\,n$, construct (Art. 39) the point O' where the conic $a\,d\,Q\,R\,O$ cuts again the line $P\,n$ (Art. 8).

45. To construct the conjugate to any point, O, in the plane PQR. Draw PO' polar to O with respect to the angle mPn, and draw the straight line aO cutting Pn in h. Construct on Pn the point h' conjugate to h (Art. 44). The point O', where the straight line PO' cuts the conic $PQRh'a$ is readily constructed, and is the conjugate to O.

For the conjugate to O is the intersection of the conjugates of the lines PO and aO, that is, the intersection of the line PO' and the conic through P, Q, R, and the conjugates of a and h. But a is conjugate to itself, and h' is conjugate to h.

46. If Q and R are the imaginary circular points at infinity, this method of transforming curves becomes essentially that of reciprocal radii vectores.

For in accordance with the constructions of Art. 42, the point m (Fig. 6) is at infinity on the line Pa, and ad is bisected in P. The conic C is any circle, and the polar of m with respect to C is a diameter perpendicular to ad. Hence Pn, drawn to meet this polar at infinity, is perpendicular to ad. Any circle through a and d will cut Pn in two points, h and h', conjugate to each other (8), and



$$Pa^2 = -Ph \cdot Ph'.$$

Again, the polar to any point, O, with respect to the right angle nPm, makes the angle $O'Pa = OPa$ (*Géom. Sup.*, Art. 80), and if Oh be drawn and h' be conjugate to h, the point O', where the circle at aPh' cuts PO', is the conjugate of O (Art. 45). Now the angle hah' is a right angle, being in a semicircle, and hence $Pha = Pah' = h'O'P$. But the angle $h'PO' = hPO$, and therefore the triangles PQh and and $PO'h'$ are similar, and $PO \cdot PO = hP \cdot Ph' = Pa^2$.

If then the figure containing O' be revolved 180° about ad as an axis, O and O' will be on the same straight line through P, and $PO.PO' = a$ constant. But these are the relations of two conjugate points in the transformation by reciprocal radii vectores.

47. If we have given two imaginary points, and Q and R are also imaginary, we can construct a line and a conic through the two conjugate imaginary points, thus:—

Construct two points of the conic through P, Q, R, and the two imaginary points (Art. 40). Find the conjugates to these two points (Art. 45), and the line through them is the required line. The conic through P, Q, R, and two points conjugate to any two points of the line that passes through the given points, is the required conic.

48. All these constructions of conjugate points, where either the base points or the given points, or both, are imaginary, can be performed by the straight edge alone without the use of compasses.

49. This method of construction not only applies to curves that shall pass through given points, but in many cases conditions of tangency may also be imposed; for when two curves are tangent to each other, the curves conjugate to them touch each other (Art. 12). Also, if one condition is that there shall be a cusp at a given point, this may sometimes be introduced. For if a curve touch an axis, the conjugate curve may easily be shown to have a cusp at the opposite base point.

50. To avoid repetition, the following notation will be used.

A curve through *given* points will be indicated by letters representing those points. Exponents indicate corresponding multiple points. Thus the cubic, having a double point at the given point a, and passing through the six given points b, c, d, e, f, and g, will be called the cubic $a^2 b c d e f g$.

When dots follow the letters, they denote the number of condi-

tions which the curve can be made to satisfy, besides passing through the given points indicated by the letters. Thus the conic $abc..$ indicates the conic that passes through the given points a, b, and c, and satisfies two other conditions.

51. A few examples of the construction of tangent curves will be given. They can be multiplied indefinitely.

The theory of conics affords means of drawing a line through a given point to touch a conic that passes through five given points; hence may be described: —

a line a. to touch the cubic ab^2cdefg;
the biquadratic $a^2b^2c^2defgh$;
a conic $abcd$. to touch the line ef;
the conic $abefg$;
the cubic $a^2bcefgh$;
the biquadratic $a^2b^2c^2efghk$.

For the construction may be made to depend on that of a line through a given point, tangent to a conic through five given points, by taking a, b, and c as base points in each case. There are two solutions, for there are two tangents from a point to a conic.

52. A line can be drawn to touch two conics when a common chord of the conics is given. For this by projection may depend on the problem of drawing a line tangent to two given circles, which is an easily solved problem.

Hence may be described a conic $abc..$ to touch, —
the line de, and the line fg;
the conic $abfgh$;
the cubic $a^2bcdfgh$;
the biquadratic $a^2b^2c^2defgh$;
the conic $abdef$ and the conic $abghk$ (or $acdgh$);
the cubic $a^2bcdghk$ (or abc^2degh);
the biquadratic $a^2b^2c^2defgh$;

the cubic $a^2bcdefg$ and the cubic $a^2bcdhkl$ (or ab^2cdehk);
the biquadratic $a^2b^2c^2dehkl$;
the biquadratics $a^2b^2c^2defgh$ and $a^2b^2c^2deklm$.

53. The construction by points of a conic that shall touch three given straight lines and pass through two given points is possible. Hence can be constructed the biquadratic $a^2b^2c^2de\ldots$ to touch the three conics $abcfg$, $abchk$, and $abclm$. One or more of these conics may be replaced by straight lines through double points of the biquadratic, or the curve may be required to have cusps at one or more of the double points. Again, if b, c, and d are in a straight line, the biquadratic is replaced by the cubic $a^2bce\ldots$ If in addition the points a, c, and e are in a straight line, the biquadratic becomes the conic $ab\ldots$ There are four solutions.

54. The most general problem of constructing a conic to pass through given points, touch given straight lines, or given conics, is to construct a conic to touch five given conics. For a point may be considered as an indefinitely small conic, or a conic may be considered as a straight line or as two straight lines. The points may be imaginary, as when the conic is required to be a circle. This general problem is doubtless impossible with the rule and compasses.

Most of the special cases hitherto solved are included in the following problem: *To construct a conic $ab\ldots$ to touch the three conics $abcde$, $abfgh$,* and $abklm$, together with the problem reciprocally polar to it, viz. *Given three conics, each of which touches the same two straight lines, to describe a conic tangent to the two lines and three conics.* Solutions of these problems are given by various writers. We mention only PONCELET (*Propriétés Projectives des Figures*, Art. 406) and CHASLES (GERGONNE'S *Annales*, XVIII. 293). There are eight solutions to each problem.

55. Upon the solution of the first of these constructions depends the construction of the biquadratic $a^2b^2c^2de\ldots$ to touch three biquadratics, $a^2b^2c^2defgh$, $a^2b^2c^2deklm$, and $a^2b^2c^2denop$.

By supposing some of these points on straight lines the problem may be changed in many ways. Thus, if b, c, and d are on one straight line, the construction is that of a cubic $a^2bce\ldots$ to touch three cubics.

If a, b, and f are on one straight line, b, c, and g on another, and a, c, and h on a third, then the biquadratic $a^2b^2c^2defgh$ becomes the line de.

And so in various ways the curves may become cubics, conics, or right lines.

56. The general solution corresponding to that of the second of the problems in Art. 54 is quite complicated. The following special case may, however, deserve mention.

A conic can be drawn to touch five given lines; hence can be described a biquadratic, $a^2b^2c^2\ldots\ldots$, to touch five conics, $abcde$, $abcfg$, $abchk$, $abclm$, and $abcno$. For any or all of these conics can be substituted straight lines, each passing through a double point, though not more than two lines can pass through the same double point. Instead of one, two, or three of the lines, or conics, there may be required as many cusps at the double points. There is but one solution, and the construction can be performed by the straight edge alone.

57. PONCELET shows (*Prop. Proj.*, Art. 444) that a conic may be constructed that shall have a double contact with a given conic, and that shall touch given lines, or pass through given points, the lines and points to be three in number. The number of solutions in each case is four. These constructions, together with those of the two special cases in the 54th Art., may be made to include all the solutions that I now recollect to have met with of the general problem, *To draw a conic tangent to five given conics.*

58. Since the conic $def\ldots$ can be drawn (Art. 57) to touch doubly a given conic, it is possible to construct the biquadratic $a^2b^2c^2def\ldots$ to touch doubly the biquadratic $a^2b^2c^2ghklm$.

59. If d, e, and f are on the lines ab, ac, and bc severally, then the biquadratic $a^2 b^2 c^2 def..$ is replaced by a straight line; for the biquadratic becomes four straight lines, of which three, ab, ac, and bc, meet $a^2 b^2 c^2 ghklm$ only in a, b, and c, so that the points of contact must be on the fourth line. Hence can be constructed, with the rule and compasses, *the four double tangents of the biquadratic* $a^2 b^2 c^2 ghklm$.

60. Again, if g, h, and k are taken on the lines ab, ac, and bc, the biquadratic $a^2 b^2 c^2 ghklm$ is replaced by the line lm; hence can be constructed four biquadratics $a^2 b^2 c^2 def..$, of which a given line, lm, shall be a double tangent. In like manner can other special cases of the problem of the 58th Art. be obtained.

61. Since a conic, $de...$, can be drawn to touch a line, fg, and to to touch doubly the conic $hklmn$, it is possible to construct the biquadratic $a^2 b^2 c^2 de...$ to touch the conic $abcfg$, and to touch doubly the biquadratic $a^2 b^2 c^2 hklmn$. A score of special cases may be obtained by supposing these curves to break up into lines. If f is on ab, the conic $abcfg$ is replaced by the line ag. If g is also on ab, the conic becomes the straight line $bcfg$, together with any line through a. The condition of touching such a conic is evidently equivalent to a cusp at c.

62. Because a conic, $d....$, can be drawn to touch doubly a given conic, $efghk$, and singly two straight lines, lm and no, therefore can be drawn a biquadratic, $a^2 b^2 c^2 d....$, to touch doubly the biquadratic $a^2 b^2 c^2 efghk$, and to touch singly the conics $abclm$ and $abcno$. In this case the biquadratic $a^2 b^2 c^2 d....$ may be required to be the cubic $a^2 bc....$, the given biquadratic may be a cubic, or a conic, or a right line, or the two conics may be replaced by lines through double points, or by cusps at those points.

63. Again, a conic can be described to touch three given straight lines, and touch doubly a conic; hence can be constructed a biquadratic, $a^2 b^2 c^2.....$, to touch three conics, $abcde$, $abcfg$, and $abchk$, and

to touch doubly the biquadratic $a^2 b^2 c^2 l m n o p$. A special case is the construction of the biquadratic $a^2 b^2 c^2 \ldots\ldots$ to touch three given straight lines, and have a fourth given straight line as a double tangent.

64. A faisceau of cubics, all of which pass through eight points (and therefore through a ninth), cuts a right line through one of these points in a series of couples of points in involution (CHASLES, in *Comptes Rendus*, XLI., Dec. 24). The double points of this involution are easily constructed, and are the points of tangency of two cubics passing through the eight points and touching the given line; hence can be constructed a cubic, $a b c d e f g h.$, to touch a right line $h k$. Therefore can be constructed a cubic, $a b c d e f g h.$, to touch a conic, $a b c h k$; also a biquadratic, $a^2 b^2 c d e f g h l.$, to touch the conic $a b c h k$, or the line $a k$, or the line $h l$, or the curve may be required to have a cusp at a.

65. In like manner the construction of the biquadratic through fourteen single points enables us to construct the biquadratic through thirteen points, and tangent to a right line through two of the points. This by transformation, as in previous articles, enables us to construct various curves; but they are all of a higher degree than the fourth.

66. It is hardly necessary to add, that in all these cases the points may be imaginary by couples, or, instead of passing through two given points, the curve may be required to touch a line in a given point.

Tangencies of a higher order, with given curves at given points, may in most instances also replace a corresponding number of points.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS IN No. V., Vol. III.

THE first prize is awarded to J. COMFORT, Princeton College, N. J.
The second prize is awarded to WM. H. EVANS, Marietta College, O.
The third prize is awarded to A. STANLEY, London, England.
The fourth prize is awarded to M. C. COREY, Exeter, N. H.

PRIZE SOLUTION OF PROBLEM I.

BY ALL THE COMPETITORS.

Find a point without two concentric circles, from which if two tangents be drawn to the circles the one shall be the double of the other.

Let R and r be the radii of the two circles, x the distance of the required point from the common centre; then $x+R$ and $x+r$ will be the secants, and $x-R$ and $x-r$ the external segments; and since the tangent is a mean proportional between the secant and its external segment, we have $2\sqrt{(x^2-R^2)} = \sqrt{(x^2-r^2)}$, whence

$$x = \sqrt{\left(\frac{4R^2-r^2}{3}\right)}.$$

PRIZE SOLUTION OF PROBLEM II.

By A. STANLEY, London, England.

Any whole number and its fifth power, when divided by 30, leave the same remainder; also, the number and its seventh power, when divided by 42, leave the same remainder.

Whatever whole number greater than 2 (when $n=2$, the truth of the theorem is evident) n may represent, the product

$$(1) \qquad (n-2)(n-1)\,n\,(n+1)(n+2)$$

is exactly divisible by the product $1.2.3.4.5$, since the quotient represents the number of combinations of $n+2$ things in groups of 5. Similarly, $(n-1)\,n\,(n+1)$ is divisible by $1.2.3$, so that the

product (1), as well as the product $5(n-1)n(n+1)$, and hence also their sum, must be divisible by $1 . 2 . 3 . 5 = 30$; but this sum is clearly equal to $(n-1)n(n+1)(n^2+1) = n^5 - n$; whence we conclude that n^5 and n give the same remainder when divided by 3.

In a similar manner, it may be shown that n^7 and n give the same remainder when divided by $1 . 2 . 3 . 7 = 42$.

Prize Solution of Problem III.

There are n straight lines making, with another fixed straight line, the angles $\alpha, \beta, \gamma \ldots$; a point, P, is taken, such that the sum of the squares of the perpendiculars from it on these n lines is constant. Find the conditions that the locus of P may be a circle.

The equations of these n straight lines are

$$x_1 \cos \alpha + y_1 \sin \alpha - \delta_1 = 0, \qquad x_2 \cos \beta + y_2 \sin \beta - \delta_2 = 0, \text{ \&c.,}$$

and the n perpendiculars, dropped from $P(x, y)$ on these lines are (Salmon's Conic Sections, p. 21)

$$\pm(x \cos \alpha + y \sin \alpha - \delta_1), \qquad \pm(x \cos \beta + y \sin \beta - \delta_2), \text{ \&c.}$$

The sum of the squares of these perpendiculars must equal a constant, and in order that the locus of the equation thus formed shall be a circle, the coefficients of x^2 and y^2 must be equal, and the coefficient of xy must equal zero. These conditions readily lead to

$$\cos 2\alpha + \cos 2\beta + \cos 2\gamma + \ldots = 0, \tag{1}$$

$$\sin 2\alpha + \sin 2\beta + \sin 2\gamma + \ldots = 0. \tag{2}$$

This is essentially the solution given by all the competitors.

Prize Solution of Problem IV.

By J. Comfort, Princeton College, N. J.

If A and B represent the semi-axes of an ellipse, the altitudes of the minimum circumscribing isosceles triangles, having their vertices in the axes produced, are as $A : B$.

From the equation of the tangent $A^2yy' + B^2xx' = A^2B^2$, we get $y = \frac{B(A+x')}{\sqrt{(A^2-x'^2)}}$ when $x = -A$, and $x = \frac{A(A+x')}{x'}$ when $y = 0$. But the area of the triangle having its vertex in the transverse axis produced is $xy = \frac{AB(A+x')^2}{x'\sqrt{(A^2-x'^2)}} = u$. Let

$$u = 2\log(A+x') - \log x - \tfrac{1}{2}\log(A^2-x'^2),$$

then

$$\frac{du}{dx'} = \frac{2}{A+x'} - \frac{1}{x'} + \frac{x'}{A^2-x'^2} = \frac{2A-x'}{A^2-x'^2} - \frac{1}{x'} = 0.$$

$$\therefore x' = \frac{A}{2}, \text{ and } x = \frac{A(A+x')}{x'} = 3A.$$

Then $x' = \frac{A}{2}$, $\frac{d^2u}{dx'^2} = \frac{76}{9}A^2$, $\therefore u$ is a minimum.

In like manner, the altitude of the other triangle may be shown to be $3B$. Hence their altitudes are as $A : B$.

Prize Solution of Problem V.

By William H. Evans, Marietta College, Ohio.

Prove that, of all circular sectors having the same perimeter, the one of greatest area is that in which the circular arc is double the radius.

Let r denote the radius of a sector whose arc is $2r$. The perimeter will be $4r$, and area r^2. If the radius be diminished by any quantity, h, the perimeter remaining the same, the arc becomes $2r + 2h$, and the area $= r^2 - h^2$. But if the radius be increased by any quantity, h, then, the perimeter remaining the same, the arc becomes $2r - 2h$, and the area $= r^2 - h^2$, and in both cases the area r^2 is diminished. Hence the proposition is proved.

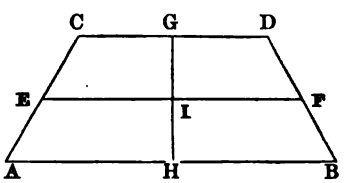
DEMONSTRATION OF THE LAW OF EQUILIBRIUM IN THE LEVER.

By Dr. Theodore Strong, Professor of Mathematics in Rutgers College, New Brunswick, N. J.

The principle on which the demonstration is based is, that for any proposed weight or power applied at any point of any straight lever of the first kind, we may always substitute two equal weights or powers, each equal to half the weight or power acting on opposite sides of the point, at equal distances from it, and in a way similar to that of the proposed weight or power.

That the principle is true, may be shown in the following manner.

Let $A\,B\,C\,D$ represent a quadrilateral, having its opposite sides, $A\,B$ and $C\,D$, parallel to each other, and having equal weights or powers acting in like manner at each of its angles. Then, if the straight line $E\,F$ bisects the opposite sides, $A\,C$ and $B\,D$, the quadrilateral will be in equilibrio on $E\,F$, regarded as an axis. For

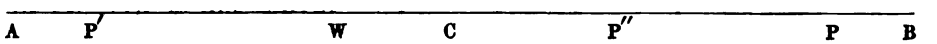


it is clear that $A\,C$ may be regarded as a straight lever of the first kind, acted on by the equal forces at A and C, which tend to turn it in opposite directions around its middle point, E, regarded as a fulcrum; consequently, since there is no reason why either of the forces should prevail over the other, they must be in equilibrio on the point. Since it may be shown, by a like reasoning, that the forces at B and D will be in equilibrio on the point F, it clearly follows that the figure is in equilibrio on the straight line $E\,F$.

If the straight line $G\,H$ bisects the parallel sides $A\,B$ and $C\,D$, a like reasoning will show the figure to be in equilibrio on the right line $G\,H$. Hence, because the forces balance each other on the right lines $E\,F$ and $G\,H$, it is evident that they must balance each other

on the point I, the common section of these lines. Consequently, since I is the middle point of GH, it may be regarded as a lever of the first kind, in equilibrio on its middle point, I; and hence the action of the forces A and B on the point H must equal the action of the forces C and D on the point G.

Supposing the distances AH and BH, as well as the forces at A and B, to be invariable, while the distances $CG = GD$ are variable; then it is manifest that the action of the forces at C and D must be invariable. Hence, if $CG = GD = 0$, it follows that the force at G is always equal to the sum of the forces at C and D, or to twice either of them; and, reciprocally, the force at C or D is half the force at G. See p. 5, Vol. I., of the *Mécanique Analytique* of LAGRANGE.



Supposing AB to be a straight lever of arbitrary length, regarded as without weight; W and P the points of the lever where the weight and power are similarly applied to it, having the fulcrum on the point on which the lever is supported between them, as in the lever of the first kind. Then, according to the law of equilibrium in the lever, *if W has the same ratio to P that PC has to WC*, the weight (W) and power (P) will balance each other on the fulcrum (C).

For suppose n to stand for any positive whole number, and that n equals the ratio of W to P or that of PC to WC; then we shall have $W = nP$, and $PC = WC \times n$. Take $WC \times (n-1)$ and set it off on each side of W to the points P' and P'' on the lever.

According to the preceding principle, we put $(n-2)P$ at W, P at P', and P'', for $W = nP$, at W; consequently, since $P'C = PC$, P at P is destroyed by the contrary action of P at P', so that, without affecting the equilibrium, we have changed $W = nP$ and P acting at the distances WC and $PC = nWC$ from the fulcrum,

into $W - 2P = (n-2)P$ and P, acting at the distances WC and $P''C = (n-2)WC$ from the fulcrum, noticing that these changes consist merely in changing n into $n-2$. It is easy to perceive that (if necessary) we may make like changes in the preceding results, and so on, until (when n is an even number) the force or weight at W vanishes, and the remaining force P is destroyed by the reaction of the fulcrum; or (when n is an odd number) until the force P at W is destroyed by the equal force P at the distance WC on the opposite side of the fulcrum, so that when n is a positive integer (as supposed) the proposed weight and power will be in equilibrio on the fulcrum.

Thus, if $n = 2$, we have $W - 2P = (n-2)P = 0$, and $P''C = (n-2)WC = 0$; and the weight at W is reduced to naught, and the power P at P'' is destroyed by the reaction of the fulcrum.

Also, if $n = 3$, the weight at W is reduced to $(n-2)P = P$, which is manifestly destroyed by the contrary action of the power P at the distance $P''C = (n-2)WC = WC$ from the fulcrum.

If m stands for any positive number, it clearly follows, from what has been done, that m times the power P acting at the extremity P of PC must balance m times the weight W acting at the distance WC from the fulcrum.

Hence, we must have $mP \times PC = mW \times WC$; and if we have in like manner the power $m'P'$ acting on the lever $P'C'$, so as to balance the weight mW by the lever WC, then, as before, we shall have $m'P' \times P'C' = mW \times WC$.

Hence, by eliminating $mW \times WC$ from these equations, we have $mP \times PC = m'P' \times P'C'$.

This equation shows that the arms of the straight lever of the first kind when in equilibrio *are inversely as the forces that are applied to them.*

Mathematical Monthly Notices.

The Mathematical Works of Isaac Barrow, D. D., Master of Trinity College, Cambridge. Edited for Trinity College by W. Whewell, D. D., Master of the College. Cambridge: Printed at the University Press. 1860. (From the Editor.)

The very best possible notice of the work before us is contained in the learned Editor's Preface, which we present to our readers entire.

"It was thought right, by Barrow's College, that a new edition of his Mathematical Works should accompany the edition of his Theological Works lately published by the University under the editorial care of Mr. Napier; and I have willingly undertaken to superintend the printing of the edition thus agreed upon. I have already, in the Preface to Vol IX. of Mr. Napier's edition, given an account of Barrow's mathematical writings; but for the sake of convenience I will here resume the subject.

"Barrow's first mathematical publication appears to have been his edition of Euclid. I have supposed, in my former Notice, that this was published in 1654, before he set out upon his travels. But the first edition is dated 1655. It is probable that he left the manuscript to be printed after his departure; for in the Preface he says that he has tried to reduce the book into a small space: 'Id quod assecutus videor, si *absentem* Typographi cura non frustretur.' This edition contains all the fifteen Books of Euclid's Elements.* I shall not insert the bulk of this publication in the present edition; but at the end of this Prefatory Notice, I will insert Barrow's Dedication and Preface to this book. The Dedication is addressed to three young men, who, from the terms in which they are spoken of, must have been, I conceive, Barrow's pupils. They were Edward Cecil, son of the Earl of Salisbury, John Knatchbul, and Francis Willoughby.† The latter was afterwards the celebrated reformer of Ichthyology, the friend of Ray, the botanist. He addresses these young men with expressions of great affection and esteem; and says that no one can know their good qualities better than himself, in virtue of the sweet habitual intercourse which he has had with them. In his Preface he speaks of the two main objects which he had in editing Euclid, to reduce the whole of the Elements into a portable volume, and to gratify those readers who prefer symbolical to verbal reasoning. He would have been satisfied, he says, with Tacquet's edition, if Tacquet had not confined it to eight books (the I., II., III., IV., V., VI., XI., and XII.), omitting the remaining seven. The symbols which he has used are mostly those of William Oughtred, 'to which most of us are accustomed.' These are the usual algebraical symbols, of which the introduction was then recent.

"But the principal contents of the present volume are the Lectures which Barrow delivered as Lucasian Professor of Mathematics; an office which he held from 1664 to 1670. And these consist of three series: the Lectiones Mathematicæ, the Lectiones Opticæ, and the Lectiones Geometricæ; the first being on the general Principles of Mathematics, the second containing propositions of Optics proved geometrically, and the third treating of properties of Curve Lines. I will make a few remarks on some of these Lectures.

"The first or Inaugural Lecture‡ contains an account of Henry Lucas, the founder of the Professorship, given in Barrow's usual rhetorical manner. He begins by referring to the tranquillity which had been restored to the nation by the cessation of the civil wars. 'The turmoil of public business being reduced to rest, and the restoration of tranquillity having given you some heart to tell and to hear news, listen to me, Academics, while I haste to tell you something strange and almost a prodigy. There has shone forth of late — What? you will say. Some dire comet, the presage of calamity, such as are seen in numbers every day (in spite of the heavens themselves) by the distorted vision of fanatics? No: but a new and benignant star, shining with a ray both true and propitious, such as has not for many years risen above the academical horizon. And it is that I may measure its magnitude, explain its motions, and interpret its presages, that I now come forwards, no vain astrologer I.' It is

* Euclidis *Elementorum* Libri XV. breviter demonstrati, Operâ Is. Barrow, Cantabrigiensis, Coll. Trin. Soc MDCLV. There are many later editions in Latin and in English.

† The entries of these names in the College Admission Book are, —

(P. 23) Edvardus Cecil, filius H D. Comitis Sar. admissus commensalis Nov. 30, 1652.

(P. 2.) Johannes Knatchbul, Cantianus, admissus commensalis May 6, 1652.

(P. 23.) Franciscus Willoughby, Warwicensis, admissus commensalis Sept. 9, 1652.

All the three being admitted as pupils of Mr. Duport.

‡ Though this Lecture is printed in the ninth volume of Mr. Napier's edition, I have reprinted it here as the first of the Lucasian Lectures; and have also ventured to repeat in this Preface some of the remarks which I made in the Preface to that volume.

by this image, and others in the same strain, that he describes LUCAS, and his good deeds towards the University. The Lucasian Professorship which he founded he endowed with the rents of an estate in Bedfordshire, amounting at present to £155. BARROW mentions the principal circumstances of LUCAS's life: stating that he had studied at St. John's College, and was then taken into the family of the EARL OF HOLLAND (who was Chancellor of the University from 1625 to 1648), as his secretary. Here, by attention to business and economy, he accumulated a considerable fortune; and continuing unmarried, resolved to make posterity his heir. He represented the University in Parliament from 1640; and as his end approached, considered and consulted with his friends how he could best promote its interests. The result was, that he determined to give encouragement to mathematical studies, which, though admired always, and especially in recent times, had hitherto enjoyed no patronage in the University. BARROW goes on to extend his praises to the two trustees of LUCAS's will: ROBERT RAWORTH, a lawyer, and THOMAS BUCK, a resident in the University, whom he refers to as well known to his hearers; 'the same whose stately presence and dignified countenance is every day before your eyes'; and to him he ascribes the suggestion of LUCAS's benefaction. These two, in conjunction with the Heads of Colleges, and especially the Vice-Chancellors of 1663 and 1664, carefully and diligently took the proper steps for carrying the bequest into effect.

"He then proceeds to speak of his own tastes, and of his purposes with regard to lectures, as I have stated in the notice of him already referred to; and ends his praise of mathematics in language which, as I have said, may remind us of the expressions of FRANCIS BACON.

"The Lectures which in the present volume next follow this Prefatory Lecture are those which were the earliest delivered, namely, in 1664, 5, and 6, but the latest published, namely, not till 1685. They treat, as I have already said, of the general principles and arrangements of mathematics, with a notice of some of the leading controversies and criticisms which had appeared on the subject, extending through twenty-three lectures, and ending with a vindication of EUCLID's Doctrine of Proportion. I have annexed to these, at the foot of the page, a brief summary of their contents, which may enable the English reader to trace the general course of the argument. They display great metaphysical subtlety, logical precision, and a large acquaintance with the literature of mathematics, both ancient and modern. As I have stated in the former account of BARROW, an English translation of these Lectures was published in 1734; but so badly executed that it cannot be of use to any one. The Lectures themselves are of interest to those who love to dwell on the metaphysical grounds of mathematical truths, but do not belong to that progressive line of mathematical speculation to which BARROW had referred in his Prefatory Lecture when he spoke of GALILEO, GASSENDI, GILBERT, MERSENNE, CARTESIUS, and others. BARROW's contributions to this kind of mathematics appear in his Lectiones Opticæ and Lectiones Geometricæ, as we shall see.

"But next after the Lectiones Mathematicæ, I have printed four Lectures, in which he proposes to himself, as he says, to expound the method by which ARCHIMEDES invented his beautiful theorems (those, namely, concerning Cones and Spheres, their Solid Contents and Surfaces). This he says he will do by reducing the steps to problems, such as ARCHIMEDES proposed to himself, and from the solution of which he deduced both his theorems and the mode of demonstrating them: whence, he says, it will appear what was the analysis which he used, and how like our modern analysis it was. It is a thought which has often suggested itself to the readers of Ancient Greek mathematicians, and especially of ARCHIMEDES, that those writers must have been led to their geometrical theorems and proofs by some methodical analytical process. Their constructions and propositions are so complex, recondite, and abstruse, that it seemed impossible that any one should be led to them by mere direct exercise of ingenuity or felicity of conjecture. We know that some modern mathematicians, particularly NEWTON, have followed this practice of discovering a proposition by analysis, and then proving it by a geometrical synthesis. BARROW has, in the four Lectures here given, proved, partly by the use of algebraical analysis, some of the most difficult of the Propositions of ARCHIMEDES concerning Spheres and Cones: for instance, that which forms the last proposition in these Lectures, that of all segments of spheres, of equal extent of surface, the hemisphere has the largest content: a proposition which might be found difficult to demonstrate by a good mathematician of the present day. And as a previous step, he has to solve this problem (Lect. XXVII. Prob. IV.): To cut a given sphere into two segments which have a given ratio. He obtains, by algebraical processes, a certain proportion which gives the point of section of the axis; and which is, he says, the very proportion to which ARCHIMEDES reduces the problem; and this, he says, shows what kind of analysis that author must have employed. For that he got at it by employing the various compositions, divisions, permutations, and inversions of proportion, in the same order in which he presents them in his text, is beyond belief. If he had done this, his lighting upon the right solution would have been rather a matter of chance than of reason or skill; and that this should have happened so constantly, is inconceivable and impossible.

"BARROW's love of the ancient Greek geometers, and his desire to abridge and simplify their demonstrations by introducing into them analysis, led him at a later period to publish an edition of the works of ARCHIMEDES, APOLLONIUS, and THEODOSIUS.*

* ARCHIMEDIS *Opera*; APOLLONII Pergæi *Conicorum* Libri quatuor; THEODOSII *Sphærica* Methodo novo illustrata et succinctè demonstrata. Per Is. BARROW, exprofessorem Lucasianum Cantab. et Societatis Regiæ Soc. 1675.

"To include this work among BARROW's Mathematical Works would have made this publication too bulky: I have printed, at the end of these prefatory remarks, the brief preface which he prefixes to it. His edition contains all the extant works of ARCHIMEDES; namely, the Two Books on the Sphere and Cylinder, agreeing in substance with the four Lectures here given; The Treatises on the Measurement of the Circle; On Spirals; On Conoids and Spheroids; On the Centres of Gravity in Plane Figures; On the Quadrature of the Parabola; On Floating Bodies; On Numbering the Sand. Also his Lemmas, which include, among other propositions concerning tangencies of circles, the proposition concerning the figure which he calls the *Arbelos*, a figure bounded by three semicircles, and named from its resemblance to a leather-cutter's knife.

"Of APOLLONIUS, this edition by BARROW contains only four Books, the only ones which have come down to us in Greek. Books V., VI., and VII. were afterwards found to exist in an Arabic translation; were brought from the East by GOLIUS, translated by ABRAHAM ECCHELLESIUS, and published by BORELLI in 1661. The Eighth Book has never been found, but has been restored conjecturally by HALLEY and by VIETA.

"The Three Books of the Spherics of THEODOSIUS contain various propositions concerning the circles of the sphere, small circles as well as great. The propositions here contained are the basis of Spherical Trigonometry; but of course the ancient Greek geometer does not attempt the solution of spherical triangles.

"The Opticæ Lectiones, as I have mentioned in my former account of BARROW, are noticed by historians of mathematics as an important work. In his Optical Speculations, says MONTUCLA, BARROW quitted the beaten track, and discussed questions hitherto imperfectly treated; as the theory of the foci of spherical surfaces and lenses, the apparent places of images, and the like. He also explains the rainbow, simplifying CARTESIUS's calculations; and, seeing that the Cartesian explanation of the colors is not satisfactory, proposes one of his own (Lect. XII. Art. XVI.). It remained for NEWTON to give the true explanation. In the Epistle to the Reader, he states that ISAAC NEWTON had revised and corrected the copy, and added matter of his own; and that COLLINS (who, he says, may be called the *Mersenne* of England, for his merits in promoting mathematical science both by his own labors and those of others) had superintended the edition with great attention. The mathematical reader will recollect that MERSENNE was a correspondent of most of the scientific men of his time, and a centre of correspondence among them.

"The Geometricæ Lectiones are full of curious methods of determining the areas and tangents of curves, many of which are very close anticipations of NEWTON's methods. The most noted of these is the method of drawing tangents to curves, given in Lect. X., Art. XIV. This method is justly held to be an anticipation of the Differential Calculus, and to approach very near to it. It will be best explained by taking BARROW's first example, which is this: —

"In Fig. 116, ABH is a right angle, K any point in BH; A being a fixed point, AK is joined; and in it, AM is taken equal to BK: it is required to draw a tangent to the curve AM.

"If, as in modern notation, we draw an ordinate, MP, perpendicular to the line of abscissas, AB, and call AP and PM, x and y respectively, AB being $= r$, it is evident that $BK = \frac{ry}{x}$, and $AM = \sqrt{x^2 + y^2}$; whence the equation of the curve is $x^2 + y^2 = \frac{r^2y^2}{x^2}$; or, $x^4 + x^2y^2 = r^2y^2$. And if we differentiate this we obtain, for the subtangent PT,

$$\frac{y\,dx}{dy} = \frac{r^2y^2 - x^2y^2}{2x^3 + xy^2}.$$

"BARROW's mode of proceeding is this (he uses p and m for x and y, which I shall alter so as to fall in with modern notation):

"Take an ordinate and abscissa near to x and y, and let these be $x - e$ and $y - a$, e and a being small. Therefore

$$(x-e)^2 + (y-a)^2 = AQ^2 + QN^2 = AN^2 = BL^2 = x^2 - 2xe + e^2 + y^2 - 2ay + a^2.$$

"But $$AQ : QN :: AB : BL;$$

that is, $$x - e : y - a :: r : BL,$$

whence $$BL^2 = \frac{r^2y^2 + r^2a^2 - 2r^2ya}{x^2 + e^2 - 2xe};$$

and, *rejecting superfluous terms* (1), in these values of BL^2, they are

$$= \frac{r^2y^2 - 2r^2ya}{x^2 - 2xe} \quad \text{and} \quad x^2 - 2xe + y^2 - 2ay.$$

"And, equating these, and multiplying up,

$$r^2y^2 - 2r^2ya = x^4 - 2x^3e + x^2y^2 - 2x^2ya - 2x^3e + 4x^2e^2 - 2xy^2e + 4xyae.$$

"That is, *rejecting the terms which our rule rejects* (2),

$$-2r^2ya = -4x^2e - 2x^2ya - 2xy^2e,$$

or
$$r^2ya - x^2ya = 2x^2e + xy^2e;$$

or, putting the ordinate y and the subtangent t for a and e, (since they are in the same proportion as those lines,)

$$r^2y^2 - x^2y^2 = 2x^2t + xy^2t,$$

whence
$$\frac{r^2y^2 - x^2y^2}{2x^2 + xy^2} = t = PT.$$

"The peculiarity of the method consists in the steps which I have marked (1) and (2); that is, the rejection of superfluous terms. And the rule given by BARROW is this: —

"After constituting the equation to the curve, put $x - a$ and $y - e$ for the ordinates x and y: expand, and reject all the terms in which there is no a or e (for they destroy each other by the nature of the curve); reject all the terms in which a or e are above the first power, or are multiplied together (for they are of no value compared with the rest, as being infinitely small): then put y for a, and t, the subtangent, for e; and PT is found.

"It is plain that the terms thus retained are the terms involving the first powers of a and e, when, in the equation to the curve, $x - a$ and $y - e$ are put for x and y, and the equation is expanded. But the ratio of the coefficients of these terms is the ratio of dx to dy in the differential calculus: and hence the substantial identity of the two methods is evident. What remained was, to devise a notation, and to assign general rules of obtaining those coefficients.

"BARROW applies this methed to the following curves (I use the modern notation for the co-ordinates): —

"Ex. II. The curve $x^3 + y^3 = r^3$.

"Ex. III. The curve $x^3 + y^3 = rxy$, which it appears was called *La Galande*.

"Ex. IV. *The Quadratrix*, of which the equation is

$$y = \overline{r - x} \tan \frac{\pi x}{2r}.$$

"Ex. V. The curve in which the abscissa being equal to an arc of a circle, the ordinate is equal to its trigonometrical tangent:

$$y = r \tan \frac{\pi x}{2r}.$$

"Also we may regard BARROW's mode of finding the areas of curves by comparing them with the sum of the inscribed and circumscribed parallelograms (Lect. XII., Append. II., Fig. 175, 176), as leading the way to NEWTON's method of doing the same, given in the First Section of the Principia.

"I have, for the most part, retained BARROW's notation, with slight alterations where it would have been likely to mislead a modern reader. Thus where we write $A : B$, he writes $A . B$; and where we write $A > B$, he writes $A \sqsubset B$. I have retained Aq for A^2, $A\,cub$ for A^3, Aqq for A^4, and the like.

"It is a matter of labor and difficulty for a reader in these days to follow out the complex constructions and reasonings of a mathematician of BARROW's time; and I do not pretend that I have in all cases gone through them to my satisfaction. I have however, in several cases, endeavored to assist my reader to follow these demonstrations with moderate trouble."

Elements of the Differential and Integral Calculus. Prepared by ALBERT E. CHURCH., LL. D., Professor of Mathematics in the U. S. Military Academy. Revised Edition, containing the Elements of the Calculus of Variations. New York: Published by A. S. Barnes and Burr, 51 and 53 John Street. 1861.

The earlier editions of this work are so well known to teachers and students, that an extended notice would seem unnecessary. The author says: "An experience of more than twenty-five years in teaching large classes in the United States Military Academy has afforded the author of the following pages unusual opportunities to become familiar with the difficulties encountered by most pupils in the study of the Differential and Integral Calculus." A careful examination of this edition must convince the experienced teacher that it realizes all the essential requisites of a good text-book. The demonstrations are clear and rigorous, the solutions are carried out in sufficient detail to enable the student to grasp the principles involved, the

text is written with care and accuracy, and all the parts, including the Calculus of Variations, are treated as fully as would be proper in an elementary work. It is possible that some teachers may find the number of problems too small; but this want is easily supplied by the excellent collections of examples already well known. But, as in the United States Military Academy, an immediate application of the subject to professional details renders a large number of examples unnecessary; and this consideration has probably guided the author in the number selected.

The earlier part of the volume is based on the doctrine of limits; but on page 127 the infinitesimal method is introduced, and retained throughout the volume. This certainly gives rise to the suggestion, whether it would not have been better to have adopted the latter method at the beginning; but this is, without doubt, one of the points which has received the author's particular attention, and should not, therefore, be hastily condemned. We close by expressing the conviction that teachers will find this edition well adapted to the wants of beginners, and in most respects an excellent text-book.

Elementary and Practical Arithmetic; in which have been attempted various Improvements in Arrangement and Nomenclature, and in the Means of thorough Discipline in the Principles and Applications of the Science. New York: Pratt, Oakley, & Co., 21 Murray Street. pp. 291.

Arithmetic for High Schools; containing the Elementary and Higher Principles and Applications of the Science. New York: Pratt, Oakley, & Co. pp. 336.

Elementary and Practical Algebra; in which have been attempted Improvements in General Arrangement and Exposition. New York: Pratt, Oakley, & Co. pp. 220.

Algebra for High Schools and Colleges; containing a Systematic Exposition and Application of the Elementary and Higher Principles of the Science. New York: Pratt, Oakley, & Co. pp. 306.

Elements of Geometry and Mensuration. New York: Pratt, Oakley, & Co. pp. 237.

Elements of Trigonometry, Plane and Spherical; with Applications to Heights and Distances, the Areas of Polygons, Surveying, Navigation, and the Solution of Astronomical Problems. New York: Pratt, Oakley, & Co. pp. 292.

This series of text-books has been prepared by JAMES B. DODD, A. M., Professor of Mathematics and Natural Philosophy in Transylvania University. Our want of space prevents a detailed notice of each of the above volumes, and we shall therefore confine ourselves to some general remarks on some of the more striking features of the series.

In the first place, the Arithmetics are precisely the same as far as page 261, and the Algebras as far as page 220. The Elementary Arithmetic, in addition, contains thirty pages on geometrical definitions, and rules for the mensuration of geometrical figures; the High-School Arithmetic, in addition, the Progressions, Compound Interest, Annuities, Permutations, Combinations, Exchange, &c. In addition to the 220 pages comprising the Elementary Algebra, the High School Algebra contains four chapters, viz. General Theory of Equations, Continued Fractions, Logarithms and their Applications, Infinite and Infinitesimal Quantities, and Infinite Series; together with a Table of Six-place Logarithms from 1 to 10,000.

This kind of gradation in a series of books on the same subject has, it seems to us, some decided advantages. We have examined the Arithmetics and Algebras with care, and while there are some points in which we think they could be improved, they are by no means inferior books. Teachers will find that they have not been ruined by any attempts at originality to which the author was unequal, but his aim has clearly been to present the subjects, both with respect to style and arrangement, in the clearest and simplest manner.

While the Geometry as a whole impresses us favorably, we are especially pleased with

Book X., on the Maxima and Minima of Geometrical Figures, as a valuable improvement upon the works on geometry commonly used in our schools. This book will prove an excellent test of the student's appreciation of the true spirit of the subject, and be a fitting close of his studies in elementary geometry.

The Trigonometry contains, in our opinion, one serious defect, in making the values of the trigonometrical functions depend upon the length of the radius, when they do in fact depend solely upon the magnitude of the angle to which they relate. This method has been abandoned in all the best modern text-books, both at home and abroad, not only as unphilosophical, but as being far inferior for the simple purpose of instruction, and especially embarrassing to the student in reading the modern works on the higher analysis.

We are, however, especially pleased with the method and clearness with which all the applications are treated; and, as an example, would call the reader's attention to the treatment of Division of Land, or of Polygons in general.

Finally, we say that teachers, in deciding upon the adoption of a series of text-books upon the elementary mathematics, will do their pupils, as well as the author, great injustice, if they fail to give this series a fair and impartial consideration.

The Quadrature of the Circle. Correspondence between an Eminent Mathematician and JAMES SMITH, Esq., (Member of the Mersey Docks and Harbour Board,) Author of "The Question, Are there any Commensurable Relations between a Circle and other geometrical Figures? answered by a Member of the British Association for the Advancement of Science." "*Strike, but Hear!*" London: Simpkin, Marshall, & Co., Stationers' Hall Court. Edinburgh: Oliver and Boyd. 1861. pp. 200.

This is the third instance which has come to our knowledge, within the past couple of years, of an attempt to "Square the Circle." In the other cases, the authors plead guilty, in less pretentious volumes, to all of JAMES SMITH's impenetrable stupidity; but they failed in the possession of that sublime impudence which makes JAMES SMITH a character not altogether devoid of interest, and mitigates in some degree the regret that he should be embalmed in so fine a specimen of typography.

It seems that the author issued a pamphlet bearing the title quoted above, and that an amiable gentleman of London attempted to point out the fallacy into which he had fallen; but soon found the attempt useless, as the author would not even admit the evidence of his senses, that by actual measurement his ratio of $3\frac{1}{8}$ was not the true one. "Eminent Mathematician," as JAMES SMITH styles his correspondent, declines, in a remarkably forbearing and courteous manner, after finding that he had "caught a Tartar," to continue the correspondence. But his horror may easily be conceived on receiving the letter which hinted at his unsought immortality. We give his reply, with JAMES SMITH's matchless answer.

"EMINENT MATHEMATICIAN TO JAMES SMITH, ESQ.

"*London*, 13th December, 1860.

"SIR, — I have received your letter of the 30th Nov., delivered here on the 11th December.

"With reference to the words in your letter, 'previously to publishing the entire correspondence,' I must be allowed to make a remark.

"My letters to you were written in the sincere conviction that I was writing to one earnestly engaged in the search after truth, and my observations were confined to the pointing out to him how he might convince himself that he was altogether wrong. My letters were not intended for publication, and I protest against their being published, for I do not wish to be gibbeted to the world as having been foolish enough to enter upon, what I feel now to have been, a ridiculous enterprise.

"Therefore, I must desire that my name may not be used.

"I am, Sir, obediently yours,

"E. M.

"JAMES SMITH, ESQ. TO EMINENT MATHEMATICIAN.

"*Barkeley House, Seaforth, Liverpool*, 15th Dec., 1860.

"SIR, — I have the honor to acknowledge the receipt of your favor of the 13th inst.

"You object to my publishing our correspondence on the 'The Quadrature of the Circle,' and give your reasons for it.

"You first say, '*Your letters to me were written in the sincere conviction that you were writing to one earnestly engaged in the search after truth.*'

"May I request you to point out a paragraph throughout our correspondence, from which you can reasonably draw an opposite conclusion?

"You then say, '*Your observations were confined to pointing out to me how I might convince myself that I am altogether wrong.*'

"I am willing to admit that this was your intention, and if in our correspondence you have succeeded in accomplishing what you proposed to yourself, may I ask, Of what have you to be ashamed if the correspondence be published?

"You next say, '*Your letters were not intended for publication.*'

"I can as truly say, that, to a certain point in our correspondence, neither were mine.

"You then protest against your letters being published, and give as your reason, '*That you do not wish to be gibbeted to the world as having been foolish enough to enter upon, what you feel now to have been, a ridiculous enterprise.*'

"This sentence is capable of two interpretations. It may mean, that in an evil moment you were foolish enough to enter into a correspondence with a man of such ignorance and stupidity that no amount of legitimate argument could possibly influence his silly judgment. Or, it may mean, that you undertook a task which you find yourself unable to accomplish, and that in doing so you were foolish enough to enter upon, what you feel now to have been, a ridiculous enterprise.

"If the former be your opinion, I can see no reason why you should have the least objection to the publication of our correspondence, for in that case I should be the party gibbeted, not you.

"If the latter be your opinion, you have certainly not had the candor to distinctly avow it.

"Pray inform me which alternative you wish me to accept as your meaning, and I will then tell you what course I shall adopt, and give you my reasons for it.

"I am, Sir, yours very respectfully,
"JAMES SMITH."

Those wishing to study JAMES SMITH further must buy his book. It teaches one important lesson, which is sufficiently obvious without an explicit statement.

Editorial Items.

Solutions of problems in the January number have been received from MISS H. S. HAZELTINE, Worcester, Mass.; GEORGE W. PIERCE, Freshman Class, Harvard College; ASHER B. EVANS, Madison University, Hamilton, N. Y.; JAMES G. MARSTON, Military Institute, Franklin County, Ky.; DAVID TROWBRIDGE, Perry City, N. Y.; JOHN Q. HOLLISTER, Hamilton College, Clinton, N. Y.; W. K. LEMMON, McKendree College, Lebanon, Ill.; H. J. McMURPHY, Londonderry, N. H.; SAMUEL MANN, Athol, Mass.; WILLIAM TIMPSON, White Plains, N. Y.; MARCUS M. RHOADES, Mount Pleasant College, Huntsville, Mo.; JAMES F. ROBERSON, Grantsburgh, Ind.; J. M. TERRILL, Linneus, Mo.; E. O. GIBSON, South New Berlin, N. Y.; GEORGE B. HICKS, Cleveland, Ohio.

Solutions of problems in the February number have been received from JAMES CLARK, Wayne, Me.; BOON GRIFFIN, McKendree College, Ill.; JOHN A. WINEBRENER, Princeton College, N. J.; DAVID TROWBRIDGE; WILLIAM H. EVANS, Marietta College, Ohio; E. CANFIELD, Chester, N. J.; M. K. BOSWORTH and HIRAM L. GEAR, Marietta College, Ohio; C. H. WILLARD, Antioch, Ill.; A. G. BARKER, Waterville College, Me.; PHILO HOLCOMB, Cardington, Ohio; J. F. ROBERSON; GEORGE B. HICKS; A. STANLEY, London, England; J. COMFORT, Princeton College, N. J.; M. C. COREY, Exeter Academy, N. H.; ASHER B. EVANS, Nunda, N. Y.; E. O. GIBSON, South New Berlin, N. Y.

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THIRD PRIZE ESSAY.—THE METHOD OF PROJECTIONS.*

By ARTHUR W. WRIGHT, New Haven, Conn.

1. IF from any point in space straight lines be drawn through the angular points of any figure, and upon each of the lines so drawn any point be chosen, the lines joining such points will form a new figure bearing a certain relation to the primitive figure, and called a *projection* of it. If the ratios of the segments intercepted by the points of the two figures are equal, and of the same algebraic sign, the figures will be similar, and the ratio of their perimeters the

* The first use of Projection seems to have been made as early as, or earlier than, the time of HIPPARCHUS, who employed it in the construction of geographical charts and star maps. Long after him, CLAUDIUS PTOLEMÆUS employed for the same purpose the process which has more recently received the name of *Stereographic Projection*. The method of projections in *geometry*, however, is of comparatively recent date. The fact that the projections of the circle upon different planes are conic sections, has, indeed, long been known, and it is probable that the ancients were not wholly unacquainted with it. NEWTON, in his *Opuscula Mathematica, Philosophica, et Philologica*, Vol. I. p. 264, mentions the fact that the shadow of a circle thrown upon a plane is a conic section, and the shadow of any curve a curve of the same degree. DE LA HIRE also, who was probably the first to employ in geometrical investigations the points afterwards called by SERVOIS *poles*, and the lines named *polars* by GERGONNE, seems to have been acquainted with the fact that all the conics may be obtained from the circle by projection. Little or no use, however, was made of this principle in geometry until the early part of the seventeenth century, when DESARGUES, and, after him, PASCAL, applied it to the investigation of the properties of conic sections. DESARGUES was likewise the discoverer of the theory of

same as that of the segments of the projecting lines. The term *projection*, however, is usually limited in its application to the figure formed by joining the points in which a plane intersects the lines drawn from any point in space through the angular points of any *plane* figure, and it is in this sense that it will be employed in the present essay.

2. The point from which the projecting lines are drawn is called

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the *involution of six points*, and PASCAL of the celebrated proposition which bears his name, respecting the properties of the hexagon inscribed in a conic.

MONGE, however, was the first to give prominence to the method of projections, and made it the basis of his *Géométrie Descriptive*, in which he investigates the properties of figures of three dimensions by means of their projections upon two planes intersecting at right angles. He also established more completely the theorems in regard to poles and polars of curves of the second degree, investigated the properties of the points called by him *centres of similitude* of circles, and proved that those of three circles lie three by three in straight lines, together with the analogous proposition concerning the centres of similitude of four spheres. After MONGE, COUSINERY, in his *Géométrie Perspective*, extended the method of projections, and simplified it by making use of a single projection upon a single plane; and also BRIANCHON, applying both DESARGUES's theory of the involution of six points and the principles of projection, made many interesting discoveries, and wrote an admirable treatise, entitled *Mémoire sur les Lignes du Second Ordre*, to which PONCELET acknowledged himself greatly indebted. CARNOT developed the theory of *transversals*, and by a beautiful generalization succeeded in applying it to curved lines and surfaces.

It is to PONCELET, however, that the method of projections is most indebted for a systematic and full development, and his work, bearing the title *Traité des Propriétés Projective des Figures*, is a complete *résumé* of all that had been accomplished up to his time, together with many new and valuable propositions of his own. PONCELET introduced the *principle of continuity*, and was the first to make use of the doctrine of the *homology of figures*, as well as that of *reciprocal polars*, which he employed very elegantly and effectively. He also made use of *imaginary quantities* in geometry, a principle which has since received many important applications in the hands of M. CHASLES, who is likewise the author of many elegant theorems regarding the anharmonic ratio of four and the involution of six points, and has besides done much to extend and perfect the method of projections. We will only remark, before proceeding with the method itself, that although it has been objected that it is not in all respects sufficiently rigorous, yet its simplicity and beauty recommend it to all lovers of pure geometry, while as an instrument of investigation and discovery it is invaluable.

the *centre of projection*, and the plane upon which the projection is thrown is called the *plane of projection*.

3. The projection of a point is a point, since it is the intersection of the projecting line with the plane of projection.

4. The projection of a straight line is a straight line; for all the lines drawn from the centre of projection to such a line are necessarily in one plane, and its projection lies in the intersection of this plane with the plane of projection, and therefore is a straight line. Hence, if any number of straight or curved lines meet in one point or in any number of points, their projections will also meet in one point or in an equal number of points, as the case may be.

5. The projection of any curve is a curve of the same degree. For the degree of a curve is determined by the number of points in which it is met by a straight line. If, therefore, a curve of the nth degree is met by a straight line in n points, these points will be projected into n new points, in which the projection of the straight line meets that of the curve, which is therefore of the nth degree. Hence the projection of a circle, or any conic section, is another conic section. Also a chord of any curve meeting it in the points M and N will be projected into a chord of the projected curve meeting it in the points m and n. Now, when M and N coincide, m and n also coincide, the chord becomes a tangent, and its projection a tangent to the projected curve.

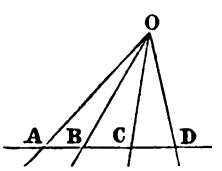
6. It is plain that any number of straight lines meeting in the same point may be projected into parallels, for, O being the centre of projection and S the common point of the given lines, if these lines are projected upon a plane parallel to a plane passing through O and S, the line OS will be parallel to the plane of projection, and hence the projection of S upon the latter will be at infinity, and the projections of the lines drawn through S will be straight lines meeting in a point at infinity, and therefore parallel. Also, if there be

any number of straight lines meeting in n points in one straight line, lm, their projections upon a plane parallel to Olm will be n systems of parallels. Hence, on the contrary, the projections of parallel lines, that is, of lines meeting in a point at infinity, are in general lines meeting in a point at a finite distance. So, also, if we have in a plane n systems of parallel straight lines, their projections will in general be n systems of straight lines, meeting in n points upon one straight line at a finite distance.

7. It is evident, from the nature of projection, that its principal application is to the relations of *position* of figures and their parts, and not to those of *absolute magnitude*. Those properties which, being true for any figure, are true also for its projections, are called *projective properties*. Hence, if we wish to prove any projective theorem in regard to any figure, we have only to establish it for one of its simplest projections, and thence can infer its truth in regard to the primitive figure.

8. If now through any point, as O (Fig. 1), there be drawn four straight lines making any angle with one another, and any line, as $ABCD$, be drawn intersecting them, putting p for the perpendicular from O on AD, we shall have

Fig. 1.



$$AB.\tfrac{1}{2}p = \tfrac{1}{2}AO.OB\sin AOB,$$

$$\therefore\quad AB = AO.OB\sin AOB \div p;$$

and, similarly,

$$BC = BO.OC\sin BOC \div p,$$
$$CD = CO.OD\sin DOC \div p,$$
$$AD = AO.OD\sin AOD \div p,$$

$$\therefore\quad \frac{AB.CD}{BC.AD} = \frac{\sin AOB\sin DOC}{\sin AOD\sin BOC} = \text{constant.}$$

Now, since this expression does not involve the distances of A, B,

C, and D from O, but only the sines of the angles at that point, the relation in question is independent of the position of AD, and therefore holds true for any of its projections. Hence, if from the points A, B, C, and D straight lines be drawn to any point in space, the segments made by those lines upon any line intersecting them will be in the same ratio as the above. This property is therefore projective.

9. The line AD is called a *transversal*, and the ratio $\frac{AB.CD}{BC.AD} = \text{const.}$, the *anharmonic* ratio of the points A, B, C, and D, in distinction from the *harmonic* ratio, which is the name given to this ratio in the particular case in which it is equal to 1. For if

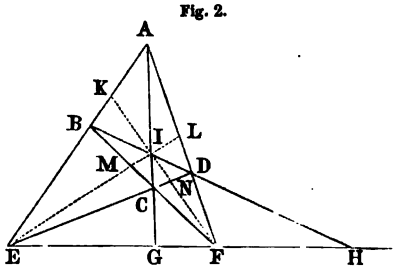
$$\frac{AB.CD}{BC.AD} = 1, \qquad \frac{AB}{BC} = \frac{AD-BD}{BD-CD} = \frac{AD}{CD},$$

or,
$$AD : CD :: AD - BD : BD - CD,$$

which is a harmonical proportion. Now, if D is at an infinite distance, AD and CD become ultimately equal, so that the anharmonic ratio becomes $\frac{AB}{BC} = \text{constant}$, and the harmonic $\frac{AB}{BC} = 1$, or $AB = BC$, and hence AC is bisected in B.

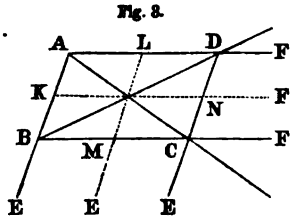
10. Again, let $ABCD$ (Fig. 2) be a quadrilateral, of which the opposite sides, AB and CD, AD and BC, meet in the points E and F respectively, forming a complete quadrilateral, as it is called. Draw the diagonals EF, BD, and AC, the two latter meeting EF in G and H, and EI and FI meeting the sides of the figure in M, L, K, and N. The figure thus formed is composed of nine lines, each divided harmonically in four points. For if the figure is projected in such a manner that the line EH passes to infinity, the lines passing through E and F become parallels (6), and

Fig. 2.



the quadrilateral $ABCD$ becomes a parallelogram, with the lines LM and KN drawn through the intersection of its diagonals parallel to the sides of the figure, as in Fig. 3. Hence each of the lines of the figure is divided in three points, of which the middle one in each case bisects the distance between the other two. The three points on each line are therefore three points of a harmonic group, of which the fourth is at infinity on the projection of EH (Fig. 2), (9). This being true for Fig. 3, is therefore true for Fig. 2, of which it is a projection. Again, the four straight lines IE, IG, IF, and IH (Fig. 2) being drawn from the same point, I, through the four harmonic points, E, C, N, and D, form a harmonic pencil, and therefore (8) cut the line EH harmonically in the four points E, G, F, and H, so that we have this theorem.

Fig. 3.



11. Each of the three diagonals of a complete quadrilateral is cut harmonically by the other two.

12. An interesting application of this theorem is made in the solution of the problem, to find, with the rule alone, the distance of an inaccessible point, as H (Fig. 2), from a given point, F. For, taking any other point, as E in the line FH, we have only to construct a complete quadrilateral whose opposite sides pass through the points E and F, and one of its diagonals through H. The other diagonal will divide the line EH harmonically (11), giving us the ratio $\frac{EG}{GF} = \frac{EH}{FH}$, which may be written $EG : GF :: EF + FH : FH$, whence we have

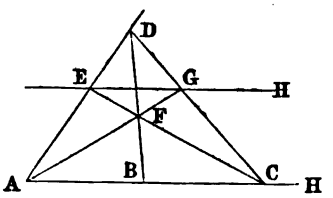
$$EG - GF : GF :: EF : FH;$$

therefore, the first three terms being known, the fourth, FH, may be found.

13. Another application may be made in the solution of the

problem, having given the middle point, B, of a finite straight line, AC, to draw with the rule alone a straight line passing through a given point, E, and parallel to AC. Join AE (Fig. 4) and EC, and from B draw a line intersecting these lines in any points, D and F; draw CD and AFG, and join EG. The figure thus formed is a complete quadrilateral, and the diagonals DF and EG cut the diagonal AC harmonically (11). But the point B bisects the line AC; therefore the fourth harmonic point, H, is at infinity (9), and hence EG is parallel to AC.

Fig. 4.



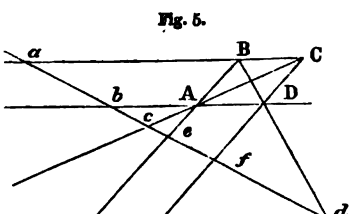
14. Again, in Fig. 3 the lines LE and KF determine eight segments upon the four sides of the parallelogram, and we have the following equation from the equality of the segments: —

$$AK.BM.CN.DL = AL.DN.CM.BK.$$

Now since for each line, AK, &c., we have, as in (8), a term of the form $AO.OK \sin AOK \div p$, it is evident from its symmetry that the equation reduces to one expressing the relation of the angles between the lines drawn from any point, O, and the points A, B, C, D, K, L, M, and N. Hence this property is projective, and therefore is true for any quadrilateral.

15. Let the transversal ad in Fig 5 meet the pairs of sides and the two diagonals of the parallelogram $ABCD$ in the points ab, ef, and cd respectively; we have, by similar triangles,

Fig. 5.



$$\frac{af}{ac} = \frac{be}{bc}, \quad \frac{ae}{ad} = \frac{bf}{bd}, \quad \frac{ea}{ed} = \frac{fb}{fd}, \text{ and } \frac{eb}{ec} = \frac{fa}{fc},$$

from which we have the equations

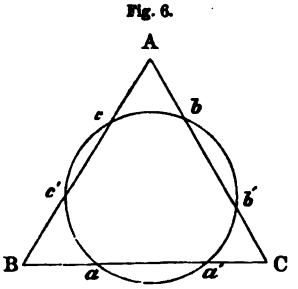
$$\frac{ae.af}{ac.ad} = \frac{be.bf}{bc.bd}, \qquad \text{and} \qquad \frac{ea.eb}{ec.ed} = \frac{fa.fb}{fc.fd}.$$

These relations are projective (14), and hence are true for the quadrilateral of which Fig. 5 may be considered as the projection. Now considering two of the opposite sides of this quadrilateral as the diagonals of the new quadrilateral formed by the two other sides and the diagonals, we have in a similar manner the equation $\frac{ca \cdot cb}{ce \cdot cf} = \frac{da \cdot db}{de \cdot df}$. Combining the three equations, we have the four following relations: —

$$cb \cdot ea \cdot fd = da \cdot ec \cdot fb \qquad cf \cdot db \cdot ea = bf \cdot ca \cdot de,$$
$$cb \cdot ed \cdot fa = da \cdot eb \cdot fc, \qquad ca \cdot df \cdot eb = db \cdot ec \cdot fa,$$

which are projective (14). The six points $a, b, c, d, e,$ and f are said to be in *involution*. Hence the theorem, any quadrilateral with its diagonals is cut by a transversal in six points in involution.

Fig. 6.



16. Again, if the sides of the triangle ABC (Fig. 6) are cut by a circle in the points a, a', b, b', c, c', we have (EUCLID, 3. 35 and 36) the equations

$$\frac{Ac \cdot Ac'}{Ab \cdot Ab} = \frac{Ba \cdot Ba'}{Bc \cdot Bc'} = \frac{Cb \cdot Cb'}{Ca \cdot Ca'} = 1,$$

which, multiplied together, give the equation

$$Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb' = Ab \cdot Ab' \cdot Bc \cdot Bc' \cdot Ca \cdot Ca',$$

which is projective in its nature (14), so that we have CARNOT'S theorem, if a conic meets the sides BC, CA, and AB of a triangle, ABC, in the points a, a', b, b', and c, c' respectively, then

$$Ac \cdot Ac' \cdot Ba \cdot Ba' \cdot Cb \cdot Cb' = Ab \cdot Ab' \cdot Bc \cdot Bc' \cdot Ca \cdot Ca'.$$

17. If the conic, instead of intersecting the sides of the triangle, is inscribed within it, the points a, b, and c (Fig. 7) coincide with

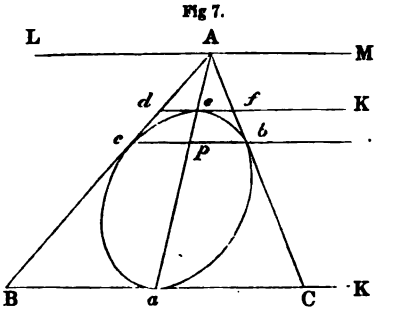
the points a'. b', and c' respectively, and the above equation becomes

$$Ac \,.\, Ba \,.\, Cb = Ab \,.\, Bc \,.\, Ca, \qquad (a)$$

which is likewise true for all projections of the figure.

18. If now the side BC (Fig. 7) is parallel to cb, which is called the *chord of contact* of AB and AC, we have, by similar triangles, $Ac : Bc :: Ab : Cb$, or $Ac \,.\, Cb = Bc \,.\, Ab$; and therefore, from equation (a), $Ba = Ca$, so that if Ap is drawn through the middle of the chord of contact cb, it will pass through the point a. In the same manner it may be shown that, if the tangent df is drawn parallel to cb, it is bisected by Apa. Now, as this must hold true for every chord of contact parallel to cb, we have the theorem, that the middle points of parallel chords of conic sections lie in the same straight line with the point of intersection of tangents drawn through their extremities, or, since this line is evidently a diameter of the curve, the chords of contact of tangents drawn through a given point are bisected by the diameter passing through that point, and are conjugate to it.

Fig 7.



19. Again, in Fig. 7, the three tangents BC, Bd, and dK may be considered as forming a triangle circumscribing the conic, and having its vertex, K, at infinity. Hence (17) $Bc \,.\, de \,.\, Ka = cd \,.\, Ba \,.\, Ke$. But $Ka = Ke$ ultimately. Therefore $Bc \,.\, de = cd \,.\, Ba$, or $\frac{Bc}{cd} = \frac{Ba}{de}$, and since BC, bc, and df are parallel, $\frac{Bc}{cd} = \frac{pa}{pe}$, and $\frac{Ba}{de} = \frac{Aa}{Ae}$; therefore $\frac{pa}{pe} = \frac{Aa}{Ae}$, which is a harmonical ratio (9). Therefore the diameter drawn through the intersecting point of two tangents to a conic is cut harmonically by the curve and the chord of contact.

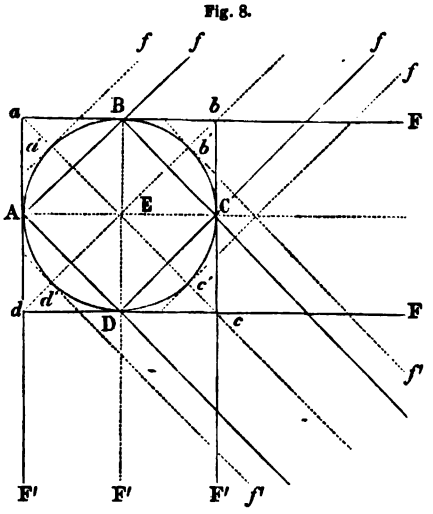
20. If now one of the extremities, a, for instance, of the diameter

of a conic passes to infinity, as is the case in the parabola, the equation $\frac{p\,a}{p\,e} = \frac{A\,a}{A\,e}$ becomes ultimately $P\,e = A\,e$, that is, in a parabola the part of the diameter lying between the summit of the circumscribed angle and the chord of contact is bisected by the curve.

21. If in Fig. 7 we draw through A a straight line to $L\,M$ parallel to $c\,b$, its direction will be conjugate to that of the diameter $A\,a$. This line is called the *polar* of the point p, which, on the other hand, is called the *pole* of the line $L\,M$. In the same manner the line $c\,b$ is called the polar of the point A.

22. If now, in Fig. 8, we suppose a rectangle, $A\,B\,C\,D$, to be inscribed in the circle, and a parallelogram, $a\,b\,c\,d$, circumscribed about it, having its points of tangency at the angles of $A\,B\,C\,D$, and the tangents $a'f$, $b'f'$, $c'f$, and $d'f'$ drawn parallel to its sides, it is evident that the opposite sides of these parallelograms meet at infinity, that their diagonals all pass through the centre of the circle, and that those of each parallelogram are parallel to the sides of the other, and hence meet them, together with the parallel tangents, at infinity. Now if we have a quadrilateral inscribed in any conic, and a straight line, $m\,n$, passing through the intersections of the opposite sides of that quadrilateral, it is evident that we may project them upon another plane in such a manner that the conic will become a circle, while the straight line $m\,n$ passes to infinity. For we may take O, the centre of projection, in such a manner that the plane $O\,m\,n$ will be parallel to the plane making a circular section with the cone formed by lines

Fig. 8.



drawn from O to all points of the conic, in which case the line will be projected to infinity (6), and the conic become a circle. Fig. 8, then, may be regarded as the projection of a conic and of its inscribed and circumscribed quadrilaterals, the angles of the former being at the points of tangency of the latter, and the opposite sides of both, with the lines from which the parallel tangents in Fig. 8 were projected, meeting upon the same straight line (6). Hence we may derive the following important conclusions.

23. The four diagonals of the two quadrilaterals meet in one point, E, which is the pole of the line containing the four points of intersection of their opposite sides.

24. The diagonals of each quadrilateral pass through the two points where the opposite sides of the other intersect, of which points each one is the pole of the line drawn through the other and the point E; that is, it is the intersecting point of the tangents drawn through the intersection of that diagonal with the conic.

25. Every straight line drawn through the point E from any point in its polar is divided harmonically by this point and the conic.

26. The point E being evidently independent of the particular position of the quadrilateral, and changing only with a change of position of its polar, we have the proposition, that if from any point in a straight line in its plane two straight lines be drawn touching any conic, the chord of contact will pass through a fixed point. Hence, on the other hand, if a secant of any conic revolve about a fixed point, the tangents at its extremities will intersect on a fixed line, which solves Pr. Prob. No. V., p. 29, Vol. I., of the Mathematical Monthly.

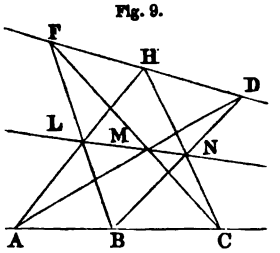
27. It is evident that the same method of proof may equally be applied to the case of the hexagon inscribed in a conic, and the circumscribed hexagon whose sides touch the conic at the angular points of the former, thus conducting us directly to the celebrated theorem of PASCAL: —

The opposite sides of every hexagon inscribed in a conic meet in three points which lie in one straight line:

Also, in every hexagon circumscribed about a conic, the three diagonals joining the opposite angles meet in a point.

28. Now, since any two straight lines lying in the same plane may be considered as a conic section,* we have the theorem, If on each of any two straight lines three points, F, H, and D (Fig. 9), A, B, and C are taken, the three pairs of lines FB, AH; FC, AD; and HC, BD intersect in three points which lie in one straight line. This proves the construction given in the Mathematical Monthly, Vol. II. p. 266, Fig. 2.

Fig. 9.



29. The above proposition (27) also enables us to solve the problem, Having given five points of any conic section, to describe that conic section by points. For if the given points are A, B, C, D, and E, join AB, BC, CD, and DE, and let AB and DE meet in m. Through m draw any straight line meeting BC and CD in l and n respectively. Join nA and lE, meeting in F. F is evidently a point of the conic (27); and in the same manner we may obtain any number of points, and thus wholly determine the curve.

30. We will now briefly consider the case in which the centre of projection is at infinity, and the projecting lines perpendicular to the plane of projection, to designate which the term *orthogonal projection* is sometimes employed. The projecting lines are evidently parallel, and we may notice the following particulars.

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* That two straight lines may be considered as a conic section is evident: 1st, from the fact that a straight line intersects two straight lines in two, and only two points (see above, Sec. 5); 2d, considering the conic as formed by the intersection of a cone and a plane, the conic will become two straight lines when the plane passes through the axis of the cone; and, 3d, because the analytical equation representing two straight lines is of the second degree (SALMON's Conic Sections, Sec. 67 *et seq.*).

31. Parallel lines are projected into parallel lines, and are diminished in a constant ratio.

32. Lines parallel to the line of intersection of the plane of a figure and that of its projection, are unaltered by projection, while those that are at right angles to this line are diminished in the ratio of the secant of the angle of inclination of the planes. So also the areas of plane figures are diminished by projection in the same ratio.

33. Now, if in any circle we draw a diameter parallel to the intersecting line of its plane with that of projection, and draw lines at right angles to this diameter meeting the circumference, these lines will all be diminished in a constant ratio (31), and hence the resulting figure will be an ellipse when the whole is projected upon another plane.

This property, therefore, affords a ready solution of the following problem:* If the three angles $A, B,$ and C of a parallelogram, $ABCD$, are in an ellipse, and the sides DA and DC meet the curve in E and F, then will a tangent to the curve at B be parallel to a line drawn from E to F; for the truth of the proposition is evident in the case of the circle from the symmetry of the figure, and hence (31, 33) it is equally true for the ellipse.†

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* Prize Problem proposed to the Sophomore Class, Yale College, July, 1857.

† The titles of some of the most valuable works, either devoted to or containing chapters upon the method of projections, are given in the following list, together with those of some of the principal journals in which may be found papers upon the same subject.

PONCELET, *Propriétés Projective des Figures;* BRIANCHON, *Mémoire sur les Lignes du Second Ordre;* MONGE, *Géométrie Descriptive;* CARNOT, *Géométrie de Position, Essai sur la Théorie des Transversals;* PLÜCKER, *Analytisch-Geometrische Entwicklungen;* CHASLES, *Aperçu Historique sur l'Origin et le Développement des Methodes en Géométrie,* also his memoirs upon *Cones* and *Spherical Conics;* SALMON, *Conic Sections, Higher Plane Curves;* QUETELET'S *Correspondance;* GERGONNE'S *Annales des Mathématiques;* LIOUVILLE'S *Journal;* GRUNERT'S *Archiv für Mathematik;* CRELLE'S *Journal;* TERQUEM *et* GERONO, *Nouvelles Annales de Mathématiques.*

MATHEMATICAL NOTES.

By MATTHEW COLLINS, A. B., Senior Moderator in Mathematics and Physics of Trinity College, Dublin.

I. *New Geometrical Theorem, and its Elementary Demonstration.*

LEMMA. — Project a straight line, $B\,D\,C$,* upon any other straight line, $b\,c$, by the perpendiculars $B\,b$, $C\,c$, and $D\,d$; then if D lie between B and C, $D\,d \times B\,C$ will be $= B\,b \times C\,D + C\,c \times B\,D$.

For, draw $b'\,D\,c'$ through D parallel to $b\,c$, meeting the perpendiculars in b' and c'; then, as triangles $B\,D\,b'$ and $C\,D\,c'$ are obviously similar, $\therefore\ C\,D \times B\,b' = B\,D \times C\,c'$; that is,

$$C\,D\,(B\,b - D\,d) = B\,D\,(D\,d - C\,c),$$

and therefore, by transposition,

$$B\,b \times C\,D + C\,c \times B\,D = D\,d \times C\,D + D\,d \times B\,D,$$

therefore $= D\,d \times B\,C$.

Now to apply this lemma. Suppose $A\,D$ bisects $\angle B\,A\,C$, then, $B\,D$, $D\,C$ and $B\,C$ are as $A\,B$, $A\,C$, and $A\,B + AC$, and therefore by the lemma ($a\,b$ being *any* straight line upon which the points A, B, C, D, O are projected), $AB \times Cc + AC \times Bb = (A\,B + AC)\,D\,d$. Again, O being *any* point in AD, $AO \times Dd + DO \times Aa = AD \times Oa$. Now, if $C\,O$ bisect $\angle A\,C\,D$, O will then be the centre of the inscribed circle, and $A\,O : O\,D = A\,C : C\,D$, or $= A\,B : B\,D$, therefore also $= A\,B + A\,C : B\,C$; so that $O\,D$, $O\,A$, and $A\,D$ are as $B\,C$, $A\,B + A\,C$, and $A\,B + AC + B\,C$, and therefore by the lemma $(A\,B + AC + BC) \times Oo = BC \times Aa + (AB + AC)\,D\,d$, and as this last term $(A\,B + AC)\,D\,d$ was already proved

$$= AB \times Cc + AC \times Bb;$$

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* The figures will be readily supplied by the reader. — ED.

hence, then,

$$AB \times Cc + AC \times Bb + BC \times Aa = (AB + AC + BC)\, Oo.$$

As a particular case of this general theorem, let us now suppose that the arbitrary line ab touches the inscribed circle, then $Oo =$ radius, and $\therefore (AB + AC + BC)\, Oo = 2 \triangle ABC$. Whence this remarkable theorem,* viz.: If each side of a *fixed* triangle described about a circle be multiplied by the perpendicular let fall from its opposite vertex upon any *variable* tangent to the circle, the algebraic sum of the products will be constant and equal to twice the area of the triangle, considering, as usual, the two perpendiculars upon one side of the tangent as positive, and the single perpendicular on the opposite side as negative.

Corollary. — When ab passes through O, then Oo vanishes; whence this other theorem, viz.: If each side of a triangle be multiplied by the perpendicular distance of the opposite vertex from *any* diameter of its inscribed circle, the greatest of the three products will be equal to the sum of the other two.

II. *New and Easy Investigation of a Remarkable Equation in Spherical Trigonometry.*

Let ABC be a spherical triangle, whose angles are denoted by A, B, C, the sides respectively opposite to them being, as usual, denoted by a, b, c; produce AC to D, making $AD = 90°$. Join BD by an arc of a great circle; then, by the first or fundamental formula in spherical trigonometry applied to $\triangle BCD$,

$$\cos BD = \cos BC \cos CD + \sin BC \sin CD \cos BCD;$$

* An entirely different geometrical demonstration of the foregoing theorem has been inserted by me in an Irish Mathematical Almanac for the Year (1861), published by Purdon Brothers, 23 Bachelor's Walk, Dublin.

that is, $= \cos a \sin b - \sin a \cos b \cos C$. Again, by applying the same fundamental formula to $\triangle A B D$, we obtain

$$\cos B D = \cos A B \cos A D + \sin A B \sin A D \cos A ;$$

that is, $= \sin c \cos A$, for $\cos A D = 0$ and $\sin A D = 1$, since $A D = 90°$; now, by equating these two values of $\cos B D$, dividing them by $\sin a$, and remembering that $\frac{\sin c}{\sin a} = \frac{\sin C}{\sin A}$, as is *easily proved geometrically* (see HYMER's Trigonometry, 3d edition, page 174), we obtain the required equation,

$$\cot a \sin b = \cot A \sin C + \cos b \cos C,$$

which, by the relations of the polar triangle, leads only to a similar equation, and to nothing new.

III. *New and Remarkable Use of the Polar Triangle in discovering Formulas in Spherical Trigonometry.*

1. By the first fundamental formula of spherical trigonometry, $\cos a = \cos b \cos c + \sin b \sin c \cos A$;

$$\therefore \cos a \cos A = \cos b \cos c \cos A + \sin b \sin c \cos^2 A ;$$

and as this formula is of course also true for the polar triangle, whose sides and angles are the supplements of the angles and sides of the primitive triangle which are respectively opposite to them,

$$\therefore \cos(\pi - A)\cos(\pi - a) = \cos(\pi - B)\cos(\pi - C)\cos(\pi - a) + \sin(\pi - B)\sin(\pi - C)\cos^2(\pi - a) ;$$

that is, $\cos A \cos a = -\cos B \cos C \cos a + \sin B \sin C \cos^2 a$. Now by equating both these values of $\cos a \cos A$, putting $1 - \sin^2 a$ for $\cos^2 a$, and remembering that $\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$, we find

(1) $\sin B \sin C - \sin b \sin c = \cos b \cos c \cos A + \cos B \cos C \cos a$;

a remarkable formula first discovered by CAGNOLI.

2. Again, by the same fundamental formula,

$$\cos b \cos c = \cos a - \sin b \sin c \cos A.$$

Multiply this by $\cos B \cos C$,

$$\therefore \cos b \cos c \cos B \cos C = \cos a \cos B \cos C - \sin b \sin c \cos A \cos B \cos C;$$

this formula being polarized as directed above, viz. by putting $\pi - A$ for a, $\pi - a$ for A, &c.; gives again

$$\cos B \cos C \cos b \cos c = -\cos A \cos b \cos c + \sin B \sin C \cos a \cos b \cos c.$$

Now by equating both these values of $\cos b \cos c \cos B \cos C$, we obtain the remarkable equation,

$$\left.\begin{matrix} \cos a \cos b \cos c \sin B \sin C \\ + \cos A \cos B \cos C \sin b \sin c \end{matrix}\right\} = \left\{\begin{matrix} \cos B \cos C \cos a \\ + \cos b \cos c \cos A, \end{matrix}\right.$$

therefore, by (1), $= \sin B \sin C - \sin b \sin c$, which gives, by transposition,

$$\sin B \sin C (1 - \cos a \cos b \cos c) = \sin b \sin c (1 + \cos A \cos B \cos C),$$

or, $\therefore \frac{1 + \cos A \cos B \cos C}{1 - \cos a \cos b \cos c} = \frac{\sin B \sin C}{\sin b \sin c}$, $\therefore = \frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}$.

Now by putting, as usual, $1 - \frac{a^2}{2r^2} + \frac{a^4}{2.3.4.r^4}$ — &c., for $\cos a$, and $\frac{a}{r} - \frac{a^3}{2.3r^3}$, &c., for $\sin a$; and then supposing r (the radius of the sphere) to be infinitely great, the foregoing equation becomes, for a plane triangle,

$$\frac{1 + \cos A \cos B \cos C}{\frac{1}{2}(a^2 + b^2 + c^2)} = \frac{\sin^2 A}{a^2} = \frac{\sin^2 B}{b^2} = \frac{\sin^2 C}{c^2}.$$

But I shall not proceed further, as these two instances sufficiently explain the method to those who have more abilities and leisure to push this curious and novel plan of proceeding further.

IV. *New and Easy Proof of* Napier's *Rules.*

By admitting only the first or fundamental formula in spherical trigonometry, viz. $\cos c = \cos a \cos b + \sin a \sin b \cos C$, *all* the cases of these useful rules may be demonstrated as follows by a *single* diagram. For the above equation being also true in the polar triangle, whose sides and angles are the supplements of the angles and sides of the primitive triangle which are respectively opposite to them,

$$\therefore \cos(\pi - C) = \cos(\pi - A)\cos(\pi - B) + \sin(\pi - A) \times \sin(\pi - B)\cos(\pi - c),$$

that is, $-\cos C = \cos A \cos B - \sin A \sin B \cos C$; and now, if the spherical triangle ABC be right-angled at C, that is, if $C = 90°$, these two equations give $\cos c = \cos a \cos b$, or $= \cot A \cot B$. Now produce CB and CA to D and F, making $CD = CF = 90°$, and as $\angle C$ is right, therefore D is pole of CF, and F is pole of CD; $\therefore AD = 90° = FB$, and hence the great circles, whose poles are A and B, must pass through D and F, and their point of intersection, E, will therefore be pole of AB; and thus each vertex of the spherical pentagon $ABDEF$ is pole of its opposite side; and as any arc drawn from the pole is perpendicular to the polar, therefore the angles at C', C'', &c., where the alternate sides meet, are all right. Moreover, as the distance of the poles of two great circles is equal to the angle at which they intersect, therefore each side of this spherical pentagon is equal to the external angle at the opposite vertex; thus, arc $DE = \angle BAC = \angle A$, arc $EF = \angle B$, $\angle AFC^{IV} = BD = 90 - a$, &c.; hence the five sides of our spherical pentagon are Napier's five circular parts (or rather their complements), viz. the hypothenuse AB, the angles A and B = the arcs DE, EF, and the *complements* of the legs $AF = 90 - b$, and $BD = 90 - a$; now the equation

$$\cos c = \cos a \cos b, \quad \text{or} \quad = \cot A \cot B,$$

gives $\cos AB = \sin BD \sin AF$, or $= \cot DE \cot EF$; and as the angles C', C'', &c. are right, therefore this relation must manifestly hold true for *every* side; that is, cos of *any* side of the pentagon is equal to the product of sines of the two adjacent or conterminous sides, or equal to the product of the cotangents of the two remote sides. Now when any two parts are adjacent in the original triangle, ABC, they are remote or separated from each other in the pentagon, and *vice versa*; thus the angle A is adjacent to AB and AC in the triangle, but $DE = \angle A$ is separated from AB and $AF (= 90 - AC)$ in the pentagon. Again, AC and BC are adjacent in the triangle, (for the right angle C lying between them is none of the five circular parts, and is therefore considered to have no effect in separating two adjacent or remote parts,) but their complements, AF and BD, are separated from each other in the pentagon by AB, &c. By attending, then, to this circumstance, we obviously get NAPIER'S rules in their *improved* form, viz.: —

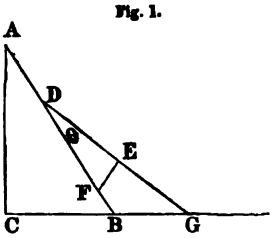
In a right-angled spherical triangle, the *cosine of the middle part is equal to the product of the cotangents of the two adjacent parts*, or *equal to the product of the sines of the two remote parts*; the five circular parts of the triangle in their consecutive order being $a' (= 90 - a)$, $b' (= 90 - b)$, A, c, B, a', b', &c., viz. the hypothenuse, the two oblique angles, and the complements of the two legs, the right angle being considered of no effect in separating any two parts, even when lying between them.

THE PENDULUM.*

By Thomas Sherwin, Principal, English High School, Boston.

Suppose a body, in virtue of gravity, to have moved along the inclined plane AB (Fig. 1) in any time to the point D, and to have acquired a final velocity per second $= V$. Suppose that at the point D the body meets another plane, DG, less elevated than AB, and making with AB the angle $BDG = \theta$. It is required to find the diminution of velocity which the body suffers at D in passing from one plane to the other.

Fig. 1.

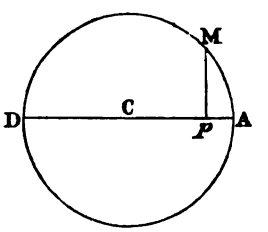


Take $DF =$ to the space over which the body would pass in one second, with the final velocity considered as uniform. DF then represents the final velocity V, or the force which produces this velocity. From F draw FE perpendicular to DG, and resolve the force DF into two forces or velocities, DE and EF. The latter is resisted by the plane DG, and DE is the force or velocity with which the body will begin to move down the plane DG. But $DE = DF \times \cos\theta = V\cos\theta$. Hence the body loses at D a portion of its velocity

$$= V - V\cos\theta = V(1 - \cos\theta) = V \text{ ver sin } \theta.$$

But if an angle or its arc is infinitely small, its sine is evidently also infinitely small, and its versed sine is an infinitely small quantity of the second order. For let AM (Fig. 2) be an infinitely small arc, Mp its sine, and Ap its versed sine. From the known properties of the circle, $Ap : pM = pM : pD$.

Fig. 2.



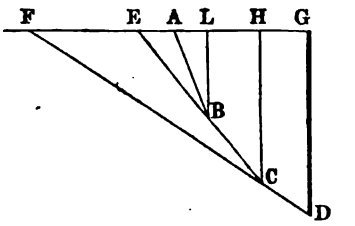
* The general mode of demonstration in this article has been taken from the French of MM. Allaize, Billy, Boudrot, and Puissant.

Now $p\,M$ is infinitely small with respect to Dp; therefore Ap is infinitely small with respect to $p\,M$; that is, it is infinitely less than an infinitely small quantity, and is hence called an *infinitely small quantity of the second order*. Hence, if in the preceding figure θ is infinitely small, V ver sin θ may be regarded as zero, and the body suffers no diminution of velocity in passing from the plane $A\,D$ to the plane $D\,G$.

A curve may be regarded as made up of an infinite number of infinitely small straight lines, any two contiguous lines differing infinitely little in inclination; hence, if a body moves down a vertical curve, its retardation at each of the successive straight lines of which the curve is conceived to be composed will be an infinitely small quantity of the second order, and the sum of these infinitely small quantities of the second order will be an infinitely small quantity of the first order and may therefore be disregarded. The body, therefore, suffers no retardation in moving down a vertical curve.

Suppose (Fig. 3) a vertical curve composed of an infinite number of straight lines, $A\,B$, $B\,C$, $C\,D$, &c. Prolong the lines $B\,C$, $C\,D$, &c. until they meet the horizontal line $F\,G$ drawn through the point A. Suppose a heavy body to have descended the plane $A\,B$ to the point B. From the laws of descent down an inclined plane, it will then have the same velocity as if it had descended the plane $E\,B$, and since it suffers no diminution of velocity in passing to the plane $B\,C$, it will have, on reaching C, the same velocity as if it had descended the plane $E\,C$. In like manner, when the body reaches D, it will have the same velocity as if it had descended the plane $F\,D$. Moreover, from the laws referred to, the body will have at the points B, C, D, &c. the same velocity as if it had fallen freely through the vertical

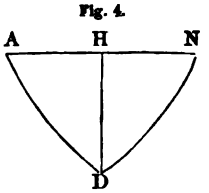
Fig. 3.



lines LB, HC, GD, &c. Hence, a body descending a vertical curve, in virtue of gravity, has at any point the same velocity as it would have acquired in falling freely through the vertical height of the curve passed over, and this velocity is independent of the nature of the curve.

When the body shall have reached the lowest point, D (Fig. 4), of the curve, it will, in consequence of inertia, begin to ascend the other branch, DN; and while it is passing over sides corresponding to those of which AD is conceived to be composed, gravity will gradually deprive it of those portions of velocity which were communicated to it in its descent along the branch AD. It will therefore ascend until it shall have reached a point, N, at the same height as the point A, from which it descended. At the point N it will begin to descend, and having reached D it will ascend the first branch to A, again to descend. The body will thus alternately move from A to N and from N to A. The movement from A to N, or from N to A, is called an *oscillation*.

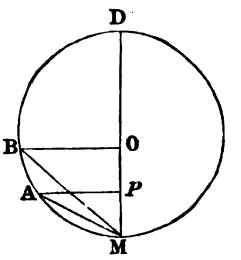
Fig. 4.



If the two branches of the curve ADN are symmetrical with respect to the vertical line HD all the corresponding elements of the two branches being equal, and being traversed with equal velocities, the times of describing them will be equal.

When the curve DBM (Fig. 5) is circular, the velocities acquired at M by two bodies that have descended the arcs AM and BM are to each other as the chords of these arcs. For, according to the laws of falling bodies and those of descent along inclined planes, these velocities are as the square roots of pM and OM, but these roots are to each other as the chords AM and BM. (See Davies's Legendre, B. IV. Prop. XXIII. Cor.)

Fig. 5.

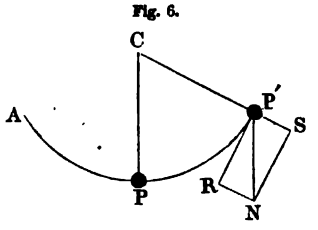


If we wish that the body, on arriving at M, should have a given velocity, V, we calculate the height $pM = \frac{V^2}{2g}$, and draw through the point p the horizontal line pA. This determines the point A, from which a body must descend along the arc AM, to acquire at M the velocity required.

2. A *simple pendulum* is a very small body of great density suspended by an exceedingly fine thread to a fixed point. Theoretically it is a mere point, supposed to be indued with weight, suspended by a thread without weight to a point about which it is perfectly free to revolve.

Suppose such a pendulum (Fig. 6) removed from its vertical position, CP, to the position CP', and then abandoned to gravity. Take $P'N$ in a vertical direction to represent the weight of the particle P. Resolve the force $P'N$ into the two forces $P'S$ in the direction of the thread, and $P'R$ in the direction of a tangent to the arc PP'. The former is resisted by the tension of the thread, and the latter produces motion down the curve. The pendulum will therefore be in the condition of a heavy body moving down the curve $P'P$, the tension of the thread corresponding to the resistance of the curve. Consequently it will move alternately from P' to A and from A to P'; and this motion would continue forever were it not for friction on the point of suspension, C, and the resistance of the air.

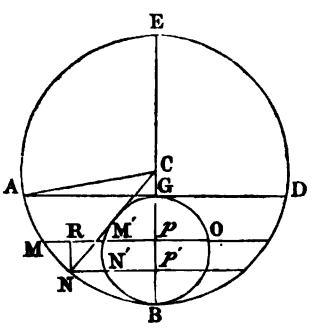
Fig. 6.



3. Considering a circular curve, EAB (Fig. 7), having its origin at E, as made up of infinitely small straight lines, let MN be one of these lines, and draw the perpendiculars Mp and Np' to the diameter passing through E. Draw also the radius CN to the point N, and RN perpendicular to Mp. The perpendiculars Mp and Np' are called *ordinates* of the curve, and pp' is called the *projection* of the

line MN upon the diameter EB. On account of the minuteness of the side MN, the radius CN may be regarded as perpendicular to it; therefore the triangles MNR and CNp' are similar, having the sides of the one perpendicular to those of the other. Hence $Np' : NR = NC : MN$, $\therefore MN = \frac{NC \times NR}{Np'} = \frac{NC \times p'p}{Np'}$. That is, any one of these infinitely small sides is equal to the product of its projection upon the diameter passing through the origin of the curve into the radius of the circle, divided by the ordinate corresponding to this side.

Fig 7.



4. It is now required to find the time of oscillation of a simple pendulum, whose length is known, through a circular arc of a very small number of degrees. Let ABD (Fig. 7) be this arc, and suppose that the pendulum, whose length is $l = CB$, starting from A, has reached the point M, and let u be the velocity it has acquired at this point. Draw the horizontal line AD, and upon GB as a diameter describe the circumference $GM'BO$. MN, as before supposed, is one of the infinitely small sides composing the curve AB, and $M'N'$ is one of those composing the curve $GM'B$. Let $Gp = x$, $Mp = y$, the small side $MN = s$, its projection pp', which is also the projection of $M'N'$, $= s'$, and the diameter GB, which is the height of the oscillation, $= d$. Also let t be the time in which the pendulum passes over MN, and T the time of an entire oscillation from A to D.

From the laws of falling bodies, and from what was proved in (1), $u = \sqrt{2gx}$; and since the minuteness of MN allows us to suppose that it is described uniformly with the velocity u, we have

$$t = \frac{MN}{u} = \frac{ls'}{y\sqrt{2gx}},$$

according to (3).

But GB, being the versed sine of a very small number of degrees, must itself be very small, and y, which is a mean proportional between pB and pE, or $d-x$ and $2l-(d-x)$, may, without sensible error, be regarded as a mean between $d-x$ and $2l$, $\therefore y = \sqrt{2l(d-x)}$. Consequently

$$t = \frac{ls'}{\sqrt{4glx(d-x)}} = \frac{\frac{1}{2}ls'}{\sqrt{glx(d-x)}} = \frac{\sqrt{l}}{\sqrt{g}} \cdot \frac{\frac{1}{2}s'}{\sqrt{x(d-x)}} = \frac{\sqrt{l}}{d\sqrt{g}} \cdot \frac{\frac{1}{2}ds'}{\sqrt{x(d-x)}}.$$

Now $\sqrt{x(d-x)} = M'p$; hence, by (3), the factor $\frac{\frac{1}{2}ds'}{\sqrt{x(d-x)}} = M'N'$. Therefore $t = \frac{\sqrt{l}}{\sqrt{g}} \cdot \frac{M'N'}{d}$; and as we should find a similar result for each of the sides composing the arc $GM'B$, it follows that the time of descent through the height of this arc, that is, the time of descent through the arc AMB, or $\frac{1}{2}T = \frac{\sqrt{l}}{\sqrt{g}} \cdot \frac{GM'B}{GB}$, and $T = \frac{\sqrt{l}}{\sqrt{g}} \cdot \frac{GM'BO}{GB}$. But the last factor $= \pi$; $\therefore T = \frac{\pi\sqrt{l}}{\sqrt{g}} = \pi\sqrt{\frac{l}{g}}$. Hence the time of the oscillation of a simple pendulum through a very small portion of the circumference of a circle is equal to the ratio of the circumference of a circle to its diameter into the square root of the quotient arising from dividing the length of the pendulum by the force of gravity.

The value of T being independent of $d = GB$, it follows that, in very small portions of a circumference, the oscillations of a pendulum are *isochronous*, or the same in duration, whether the arc be greater or less, provided it does not exceed very narrow limits. The arcs must not be greater than four or five degrees.

Remark. — It may be proved by the higher mathematics that, if a pendulum oscillates in a curve called a *cycloid*, the time of oscillation will be perfectly independent of the length of the curve.

5. The time T' of oscillation of another pendulum whose length is l', in a place where the force of gravity is g', is $T' = \pi\sqrt{\frac{l'}{g'}}$;

$$\therefore T : T' = \sqrt{\frac{l}{g}} : \sqrt{\frac{l'}{g'}}.$$

Where the force of gravity is the same, that is, in equal latitudes, $T : T' = \sqrt{l} : \sqrt{l'}$. Hence, *In the same latitude, the times of oscillation of two pendulums differing in length are to each other as the square roots of the lengths.*

If $l = l'$, $T : T' = \frac{1}{\sqrt{g}} : \frac{1}{\sqrt{g'}} = \sqrt{g'} : \sqrt{g}$. Hence, *In different latitudes, the times of oscillation of two pendulums equal in length are inversely as the square roots of the different forces of gravity.*

When $T = T'$, $\sqrt{\frac{l}{g}} = \sqrt{\frac{l'}{g'}}$; $\therefore \frac{l}{g} = \frac{l'}{g'}$, or $l : l' = g : g'$. Hence, *The lengths of two pendulums which oscillate in equal times in different places are to each other as the intensities of gravity at those places.*

If T and T' are the times of oscillation of two pendulums at the same place, l and l' their lengths, and n and n' the number of oscillations which they make in the same given time, K, we have $K = nT = n'T'$; $\therefore n : n' = T' : T$. But $T' : T = \sqrt{l'} : \sqrt{l}$; $\therefore n : n' = \sqrt{l'} : \sqrt{l}$. Hence, *The numbers of oscillations made by two pendulums of different lengths in the same time, and at the same place, are to each other inversely as the square roots of the lengths.*

6. When the pendulum and the string or rod which connects it with the point of suspension have sensible masses, as in fact they always have, it is called a *compound pendulum.* But we have seen that the time of oscillation of a simple pendulum is less, the shorter the pendulum. Hence the material particles of a compound pendulum tend to vibrate more rapidly the nearer they are to the point of support. But, being connected by cohesion, all the particles of the pendulum must oscillate in the same time. Nevertheless, the tendency to rapid motion in the particles nearer to the point of support is counteracted by the tendency to a slower motion in the remoter particles, and the slower motion which these latter tend to have is accelerated by the tendency to a rapid motion in the former. There is, however, a particle, intermediate in position, which is neither

accelerated nor retarded, but which oscillates precisely as if it were unconnected with the other particles and constituted a simple pendulum. The point occupied by this particle is called the *centre of oscillation*, and the distance of this point from the point of suspension constitutes the virtual length of the pendulum.

If in the proportion $n : n' = \sqrt{l'} : \sqrt{l}$, which gives $l' = \frac{n^2 l}{n'^2}$, l, the length of a seconds pendulum, n, the number of oscillations it makes in any time, as one minute, are known, and we find by experiment the number of oscillations, n', which a compound pendulum makes in the same time, we can easily compute l', the virtual length of this compound pendulum.

By making a pendulum whose length, l, has been very accurately determined oscillate in a vacuum during a number, b, of seconds, and observing the number of oscillations, n, which it makes in this time, we shall have the time of one oscillation by dividing b by n; that is, $T = \frac{b}{n}$. Substituting this value of T in the equation $T = \pi \sqrt{\frac{l}{g}}$, we obtain $g = \frac{\pi^2 n^2 l}{b^2}$.

By careful experiments at London, Paris, New York, and other places, the values of g have been determined, either in the manner specified above or otherwise. At New York, the approximate value of g is 385.9134 inches = 32.15945 feet. This value, substituted in the formula $T = \pi \sqrt{\frac{l}{g}}$, in which T is made equal to one second, gives $l = \frac{g}{\pi^2} = 39.1012$ inches, the length of the seconds pendulum. At Paris the length is 39.12843 inches, and at London 39.13908 inches.

THE POLYTECHNIC SCHOOL AT PARIS.

HISTORY, CONDITIONS FOR ADMISSION, COURSES OF LECTURES, GRAPHICAL WORKS, ETC.

The Polytechnic School enjoys a renown so imposing, that it is admitted as an article of faith in France, as the most efficacious mode of procuring for the state distinguished and capable subjects, whether for the public service which it supplies, or for the advancement of the exact sciences.

In order to show the relation which it bears to the government I shall give a brief sketch of its history.

Before the Revolution there existed in France the following schools of application: *Ecole des Mines*, *Ecole des Ponts et Chaussées*, *Ecole de la Marine*, *Ecole du Génie Militaire*, and *Ecole de l'Artillerie*. These special schools were almost, if not wholly, disorganized by the political difficulties which grew out of the Revolution, and at the commencement of 1794 the Committee of Public Safety felt sensibly the need of filling by competent persons the numerous vacancies which were constantly occurring.

This committee was surrounded by men distinguished for their contributions to the exact sciences. These *savans*, who were also political men, embracing with ardor the principles of the Revolution, proposed, in order to satisfy the urgent needs of the moment, to create at Paris a school destined to replace in a measure the special schools, and at the same time to give an extraordinary extension to the sciences of Mathematics, Physics, and Chemistry.

FOURCROY, a celebrated chemist, was chosen from among their number to present to the Convention the project of a law for the creation of this school, which was designated by the name of Central School of Public Works (*Ecole Centrale des Travaux Publics*).

This was a few weeks after the fall of ROBESPIERRE, and FOURCROY, with considerable address, drew the attention of the assembly to the recent triumph, in saying, that the last conspirators had formed the project, not only for suppressing the arts and sciences, in order to march to power over the ruins of human knowledge, but also for crushing all the men and removing every means useful for instruction.

The Central School of Public Works was to supply the fine corps of engineers, and it was proposed to suppress the special schools when it should be firmly established. It was also to recommence the teaching of the exact sciences, which had been suspended by the Revolution, and to diffuse the taste for these sciences, by means of the students who returned home without entering the public service.

The number of students was to be four hundred, between the ages of sixteen and twenty.

They were to be taught Descriptive Geometry, the general principles of Analysis and their application to Descriptive Geometry, to the mechanics of solids and fluids, and to the calculation of the effect of machines; also Physics, Chemistry, Design, Civil Construction, Architecture, and Fortification.

The students were to receive salaries, as the greater part of the citizens would be unable to maintain their children at Paris for three years.

FOURCROY terminated his report in these words: "The committee must tell you that the greatness of this school is worthy of the people to whom it is presented, that it will be without a model in Europe, that it will satisfy both the needs of the republic, and those of general instruction, which we have felt for the past five years, that it will diffuse wider and wider through the whole republic the taste so advantageous for the exact sciences, and that it is finally one of the most powerful means to advance equally the useful arts and the human reason."

The Convention adopted the report, and provided the means for the immediate establishment of the institution; appointing a number of professors, among whom were LAGRANGE, MONGE, HACKETTE, PRONEY, and BERTHOLLET.

The opening of the courses took place on the 24th of May, 1795. The first lecture being given by LAGRANGE, on which occasion the founder mingled with the students to listen to the illustrious geometer. Arithmetic, which was the subject of the first lesson, says PRONEY, who was present, so dry and sterile as generally treated, acquired in the hands of LAGRANGE the elegance and fertility which characterize all his writings.

Unfortunately, often in the midst of their courses, the students who were not exempt from duty in the National Guard were obliged to assist in the defence of the Convention against the attacks of the Jacobins.

It is said that in these times the students were often in the greatest need of the necessaries of life, and were only relieved by the Convention, which caused to be distributed to one hundred and fifty of their number a pound of bread per day.

The following year the Convention changed the name to Polytechnic School, and at the same time organized the schools of application, and provided that hereafter they should only receive as students the graduates of the Polytechnic School. One of the provisions of this law which regulated the conditions for admission is curious, as it shows the spirit of the times; it was as follows: "No person can present himself as a candidate for admission unless he brings a certificate from the Municipality, attesting his good conduct, and that he has constantly manifested a love of liberty and equality and a hatred of tyranny."

The school suffered occasionally from the severity of the government toward those who were not Republicans. The minister annulled the examination of two candidates for not having satisfied the law just quoted.

"This just severity," he wrote, "will recall to the students the obligation which is imposed upon them to repay by their *civism*, as well as by their work, the instruction which the country furnishes. Thus it is the formal intention of the government to accord its favors and national distinctions only to those who render themselves worthy by their devotion to the republic."

A short time subsequent, it appeared expedient to give a striking token of adhesion to the political dogmas of the dominant party. The planting of a tree of liberty was one of the most august ceremonies of this new religion. It was therefore resolved, that on the day of the opening of the courses a tree of liberty should be planted within the enclosure of the school. They tried to obtain the presence of BONAPARTE, who was then at Paris; he promised to assist, but did not come. After the addresses of the professors, the assembly repaired to the court of the laboratories, where an Athenian poplar had just been planted. The director attached to it a tricolor, and they buried at its root an account of the inauguration; they chanted couplets and recited strophes filled with fire and enthusiasm; in short, nothing was forgotten which could electrify their souls. Unfortunately it began to rain, and the republican fervor of the students did not prevent their dispersing to seek a shelter in the study-rooms, whence they saw the concluding ceremonies from the windows. This incident marred the effect, and caused the director considerable vexation.

It was on another occasion, after LAGRANGE had announced the subject of his lectures for the following year to be the Theory of Functions, designed to remove the difficulties of the Differential Calculus, that the Minister of the Interior addressed the assemblage. After a short eulogy upon the professors and the school "just raised to the first rank in public instruction," he endeavored to make the students sensible of the gratitude which they owed their country, which "had sought them from their cradles, to put them in relation with the men that Europe honored most."

"If the love of country," said he, "acts as a *sentiment* upon the rest of mankind, it is permitted to think that to the *savans* the existence of this love is geometrically demonstrated. I can say, in the language which is familiar to you, liberty is the theorem given by nature, the republic is its demonstration, and patriotism is the corollary."

The limits of this sketch must cause me to pass over with the mere mention events which had an important influence upon the character of the school. These were, the resignation of LAGRANGE and the appointment of LACROIX as his successor; the return of BONAPARTE from Egypt, and, after the memorable day at Saint Cloud, the appointment of LAPLACE, then permanent examiner, as Minister of the Interior, and the election of LEGENDRE to fill the vacancy.

I cannot, however, refrain from giving an anecdote of the teaching of MONGE. I remember, says one of his pupils, that, after having explained the most difficult part of a geometrical figure (*épure*), he usually turned to his auditory in order to discover if they had comprehended him, and if he perceived that they had not, he would recommence the same explanation with new developments, and I have seen him thus repeat three times the same explanation.

CONDITIONS FOR ADMISSION THE PRESENT YEAR, 1861.

In order to be admitted to the school, the candidate must show that he is a French subject, or that he been naturalized, that he is between the ages of sixteen and twenty, that he is free from physical defect; and he must present also a diploma of Bachelor of Science, and must submit to two examinations upon the following subjects: —

1. Geometry, Plane, Solid, and Spherical.
2. Surveying, and the use of the ordinary instruments.
3. Algebra, including Progressions, Logarithms, Derived Functions, Theory of Equations, Theory of Differences and its application to the resolution of numerical equations and decomposition of rational fractions.
4. Trigonometry, Plane and Spherical.
5. Analytic Geometry of two and three dimensions, including the discussion of the general equation of three variables.
6. Descriptive Geometry.
7. Mechanics.
8. Physics, including Hydrostatics, Hydrodynamics, Statical Electricity, and Magnetism.
9. Inorganic Chemistry.
10. The French Language.
11. The German Language, in which the student is required to know the principal rules of the grammar, to translate an easy text, reply in German to simple questions, and also to present a German theme.
12. Drawing, in which the student is required to execute, — 1. A drawing in Descriptive Geometry; 2. A drawing after a model, and shaded in India ink; 3. A design in crayon.

The first of the examinations above mentioned serves to ascertain if the candidates have the knowledge required, and the second serves to determine their rank.

The school is under military discipline, and the students are allowed to leave the establishment only twice per week.

Each student pays per year the sum of one thousand francs.

COURSES OF LECTURES. — These are eleven in number.

I. *Course of Analysis.* — This embraces during the first year the Differential and Integral Calculus with their Applications to Algebra, Geometry, and Mechanics; during the second year, Definite Integrals, Integration of Differential Equations, Application of Analysis to Mechanics and Astronomy, and Calculus of Probabilities.

II. *Course of Descriptive Geometry and Stéréotomie.* — This includes, during the first year, Axiometric Perspective, Isometric Perspective, Perspective *Cavalière*, Conical Perspective, Representation of Shades and Shadows, Developable Surfaces, Warped Surfaces, Curvature of Surfaces, the Heliçoides, and Descriptive Geometry of one Plane.

During the second year, the Application to Wood-cutting (framing and construction of staircases), Stone-cutting (rampart arches, oblique arches, conical, warped, helicoidal, and compound arches).

III. *Course of Mechanics and Machines.* — This course for the first year is divided into three parts: — 1. Cinematics; 2. Statics; 3. Calculation of the Effect of Machines.

The course for the second year consists of, — 1. Dynamics; 2. Mechanics of Fluids; 3. Mechanics considered in motion.

IV. *Course of Physics.* — During the first year, Heat and Dynamical Electricity; during the second year, Acoustics and Optics.

V. *Chemistry.* — This embraces the treatise of REGNAULT, who is one of the professors.

VI. *Astronomy and Geodesy.*

VII. *Architecture and Public Works.* — The first includes the treatise of REYNARD upon Architecture, and is divided into constructions in stone, in wood, and in iron, and the composition of edifices. The course upon Public Works consists of general notions of the construction of roads and bridges.

VIII. *Topography.*

IX. *Fortification and Military Art.*

X. *French Literature.*

XI. *The German Language.*

GRAPHICAL WORKS.

There are exercises and examples given to the students in every course, but there are regular detailed drawings required upon the following courses: —

I. *Descriptive Geometry.* — These consist of, — 1. The construction of the intersections of surfaces of the second degree; 2. The shading of cones traversed by cylinders in various ways; 3. One Plane Descriptive Geometry; 4. Linear Perspective, the drawing of a house, of an oblique bridge, and also the perspective of a *tone* with shades and shadows.

II. *Stereotomie.* — 1. Wood-cutting, including assemblages in wood, constructions of roofs, and winding staircases, 2. Stone-cutting, including construction of various doorways, oblique arches, ground and cloistered arches.

III. *Mechanics and Machines.* — The method of instruction in this branch deserves notice. A portion of a machine is placed before the students, and with pencil and paper only they are required to make a sketch of it; a measuring rule is then given them, and they are required to annex to each part of their sketch the real dimensions; afterward they are required to draw the object carefully and shade it. The students commence with elements as simple as a piston-rod with its articulations; afterwards they have toothed wheels, both cylindrical and conical, and finally a steam-engine.

IV. *Geodesy.* — The graphic works consist in making from given data a sun-dial; and 2. Delineation of maps.

V. *Military Art.* — Consisting of a trace of a front of Vauban, of Cormontaigne, and a project of attack upon a given place. The course is however very meagre, being designed to give those who are not to enter the military service hereafter some general notions which may be useful to them, — as, for instance, an engineer *des Ponts et Chaussées* may have occasion to lay out a road which borders upon a fortress, and for that it is necessary that he have some knowledge of this matter.

VI. *Architecture.* — Consisting of careful drawing of columns of the various orders, arcades, windows, porticos, and lastly the project of a public library.

VII. *Topographical Drawing.*

VIII. *Shading in India Ink.*

IX. *Portrait Drawing and Sketching of Landscapes.* — In each of the last three a model is placed before the student, and he has simply to copy it.

General Dispositions.

The lessons are usually an hour and a half in length; the last half-hour is set apart for questioning two of the students at the blackboard upon the subject of the lesson before the last.

On the table before the professor is an urn, containing one hundred and fifty balls, numbered according to the chairs in which the students sit; the professor selects at random a ball from the urn, and calls up the student occupying the chair corresponding to this number. The object of this is to find out who review the previous lesson before coming into the lecture-room.

The students are examined in each subject by the regularly appointed *repetiteurs* as often as once for every six lessons, and at the end of each half-year they are examined carefully over the whole ground by another set of men, regularly appointed, called *examinateurs*, and their rank is made up from the whole.

The students submit to military drill twice a week during the summer months.

The graduates have a choice, according to their rank, of the following services, viz.: That of *Mines*, *Ponts et Chaussées*, *Genie Maritime*, *Genie Militaire*, *Hydrographie*, *Lignes Telegraphiques*, *Commissariat de Marine*, *Tabac*, *Etat Major*, *Artillerie*, *Marine*.

To most of these services there are attached special schools, in which the students immediately enter, and pass three years in studying the specialty which they have chosen; and where there is no special school, as in the Tobacco service, the students study at the tobacco factory, and have regularly lectures and attend elsewhere to perfect themselves in the analysis of organic substances, and the higher departments of Chemistry. It is taken for granted, as the director of the studies said to me, that the students when they graduate are not fitted by their studies to enter upon any kind of practical service, and that they are only fitted in the best manner to commence the study of the application of the sciences of which they already know the principles, and hence the necessity for the schools of application; of several of these, particularly those which prepare for the civil service, I shall hereafter have occasion to speak in detail.

There are usually admitted about one hundred and seventy-five, and the number of applicants examined exceeds one thousand.

I might mention that the students are furnished with lithograph impressions of the lectures of the professors previous to their examinations, so that they are not absolutely dependent upon the notes which they take in the lecture-room, as is frequently the case in many of the other schools.

Note. — The award of the prizes for the solutions of problems in the March number, crowded out this month on account of the unexpected length of some of the articles, will be found in the next number.

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ON THE DIOPHANTINE ANALYSIS.

By Rev. A. D. Wheeler, D. D., Brunswick, Maine.

Definitions.

1. Even numbers are those which are divisible by 2 without a remainder, and which may be represented by the general expressions $2x$, $2x'$, &c.

2. Odd numbers are those which, being divided by 2, leave the remainder 1. They may be represented by the general expressions $2x+1$, $2x'+1$, &c.

Propositions.

I. The sum or difference of any two even, or of any two odd, numbers, is even.

Proof. — The expressions $2x+2x'$, $2x-2x'$,

$$(2x+1)+(2x'+1)=2x+2x'+2,$$

and $(2x+1)-(2x'+1)=2x-2x'$, are all divisible by 2 without a remainder. (Def. 1.)

II. The product of two even numbers, or of an odd number by an even, is even; but the product of two odd numbers is odd.

Proof. — The expressions $2x.2x'$, $(2x+1).2x'$ are each divisible by 2 without a remainder; but not the expression

$$(2x+1).(2x'+1)=4xx'+2x+2x'+1.$$

(Def. 1 and 2.)

Cor. 1. — The square of an even number is even, and divisible by 4; and the square of an odd number is odd, and divisible by 4 if 1 be subtracted. All possible squares are therefore contained in one or the other of these two expressions, viz. $4x$, or $4x+1$.

Cor. 2. — Hence every number which is the sum of two squares is contained in one or another of the following expressions, viz. $4x$, $4x+1$, or $4x+2$.

Cor. 3. — Therefore neither a square nor the sum of two squares is ever contained in the expression $4x+3$.

Cor. 4. — Since the expressions $4x$, $4x+1$, $4x+2$, and $4x+3$ comprehend all possible numbers, and since the expressions $4x$ and $4x+2$ are both even, while the expression $4x+3$ can never be the sum of two squares, it follows that every odd number which is the sum of two squares must be contained in the expression $4x+1$.

III. If one odd number be divided by another, the quotient will be odd; but if an even number be divided by an odd, the quotient will be even; and if it be divided by an even number, the quotient may be even or odd. No odd number can be divided by an even.

Proof. — The product of the divisor and quotient must equal the dividend, and this is impossible upon any other suppositions than those which have been made. (Prop. II.)

IV. If any number whatever be added to its square the sum will be even.

Proof. — In the expression x^2+x, if x be even, x^2 will also be even (Prop. II.); and therefore, by Prop. I., their sum will be even. If x be odd, then by the same propositions its square will be odd, and the sum of these two odd numbers will be even.

Cor. — If 1 be subtracted from any odd square, the remainder is divisible by 8. For

$$(2x+1)^2 = 4(x^2+x)+1 = 4(2x')+1 = 8x'+1.$$

V. All numbers consisting of three or more even squares are comprised in the expression $4x$.

Proof. — As all even squares are of the form $4x'^2$, $4x''^2$, $4x'''^2$, &c., their sum will be of the form $4(x'^2 + x''^2 + x''' + \text{&c.}) = 4x$. (Prop. II. Cor. 1.)

VI. All numbers consisting of three odd squares are comprised in the expression $8x + 3$; of four odd squares, in the expression $8x + 4$, and so on.

Proof. — As all odd squares are of the form $8x' + 1$ (Prop. IV. Cor.), the sum of any three of those squares will be

$$(8x' + 1) + (8x'' + 1) + (8x''' + 1) = 8x + 3;$$

the sum of any four will be

$$(8x' + 1) + (8x'' + 1) + (8x''' + 1) + (8x^{\text{IV}} + 1) = 8x + 4.$$

And so we may proceed for any larger number.

VII. The product of any two squares is a square, and likewise their quotient. The same thing is also true of the product or quotient of any even powers.

Proof. — $x^2 \cdot x'^2 = (xx')^2$; $\frac{x^2}{x'^2} = \left(\frac{x}{x'}\right)^2$. Also $x^{2n} \cdot x'^{2n} = (x^n x'^n)^2$, and $\frac{x^{2n}}{x'^{2n}} = \left(\frac{x^n}{x'^n}\right)^2$.

VIII. Any power of a quantity whose index is composed of two factors may be regarded as a quantity raised to a power denoted by either factor.

Proof. — $x^{mn} = (x^m)^n = (x^n)^m$. Thus $x^6 = x^{2 \cdot 3}$ is both a square and a cube.

IX. Let a, b, c, d represent integral numbers in geometrical proportion; and let a and b be prime to each other, and $a < c$, and by consequence $c < d$. Then will c be a multiple of a, and d a multiple of b.

Proof. — Because the numbers are proportional, we have $ad = bc$,

and $d = \frac{bc}{a}$ = an integer. But a will not divide b, being prime to it; therefore it must divide c (Ind. Anal.). For similar reasons b must divide d. In other words, c and d are respectively multiples of a and b, according to the proposition.

Cor. — By reducing c and d to their lowest terms, we shall have $a = c$ and $b = d$.

X. If each of two quantities, A and B, is the sum of two squares, their product, $A \,.\, B$, will be the sum of two squares in two different ways.

Proof. — Let $A = a^2 + b^2$, and $B = c^2 + d^2$; then their product will be

$$A \,.\, B = (a^2 + b^2)(c^2 + d^2) = (ac + bd)^2 + (ad - bc)^2. \quad (1)$$

and $$A \,.\, B = (a^2 + b^2)(c^2 + d^2) = (ac - bd)^2 + (ad + bc)^2. \quad (2)$$

Discussion.

1. If $a = 0$, both forms are reduced to one, viz. $A \,.\, B = (bd)^2 = (bc)^2$.
2. If $a = b$, both forms are reduced to one; that is,

$$A \,.\, B = (ac + ad)^2 + (ad - ac)^2 = a^2(c + d)^2 + (d - c)^2.$$

3. If $a = c$, we have the two forms, $A \,.\, B = (a^2 + bd)^2 + (ad - ab)^2$, and $A \,.\, B = (a^2 - bd)^2 + (ad + ab)^2$.
4. If $a = c$ and $b = d$, equation (1) becomes identical, and equation (2) assumes the useful and well-known form

$$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2; \quad (3)$$

that is, a square equal to the sum of two squares.

5. If $a = b$ and $c = d$, both forms become identical.
6. If $a = 1$ and $b = 1$, we have

$$A \,.\, B = 2(c^2 + d^2) = (c + d)^2 + (d - c)^2;$$

whence it appears that if any number is the sum of two squares, the same is true of the double of that number.

7. If $a^2 + b^2 = c^2 + d^2$, while the values of a, b, c, d are all different, equations (1) and (2) become the following, viz.: —

$$A \,.\, B = (a^2 + b^2)^2 = (a\,c + b\,d)^2 + (a\,d - b\,c)^2, \qquad (4)$$

and $$A \,.\, B = (a^2 + b^2)^2 = (a\,c - b\,d)^2 + (a\,d + b\,c)^2; \qquad (5)$$

that is, a square number which may be resolved into the sum of two squares in two different ways.

8. If $c^2 + d^2 = 1$, we have, in the same manner,

$$A \,.\, B = A = a^2 + b^2 = (a\,c + b\,d)^2 + (a\,d - b\,c)^2, \qquad (6)$$

and $$A = a^2 + b^2 = (a\,c - b\,d)^2 + (a\,d + b\,c)^2; \qquad (7)$$

that is, a number composed of the sum of two squares in three different ways.

XI. If A and B contain each the difference of two squares, their product will contain the difference of two squares in two different ways.

Proof. — Let $A = a^2 - b^2$, and $B = c^2 - d^2$; then

$$A \,.\, B = (a^2 - b^2)(c^2 - d^2) = (a\,c + b\,d)^2 - (a\,d - b\,c)^2, \quad (8)$$

and $$A \,.\, B = (a^2 - b^2)(c^2 - d^2) = (a\,c - b\,d)^2 - (a\,d + b\,c)^2. \quad (9)$$

Remark. — The discussion of this proposition is similar to that of the preceding, and the results are also similar, having reference to the change of signs. The most important of them are the following: —

$$A \,.\, B = A^2 = (a^2 - b^2)^2 = (a^2 + b^2)^2 - 4\,a^2\,b^2, \qquad (10)$$

and $$A \,.\, B = (a^2 - b^2)^2 = (a\,c \pm b\,d)^2 - (a\,d \mp b\,c)^2. \qquad (11)$$

XII. If each of the quantities A and B is equal to the sum of two squares, their quotient will also contain the sum of two squares in two different ways.

Proof. — Let $A = a^2 + b^2$, and $B = c^2 + d^2$; then $\frac{A}{B} = \frac{a^2 + b^2}{c^2 + d^2}$. Multiplying numerator and denominator by $c^2 + d^2$, we obtain the formulæ,

$$\frac{A}{B} = \frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2} = \frac{(ac+bd)^2+(ad-bc)^2}{(c^2+d^2)^2} = \frac{(ac+bd)^2}{(c+d^2)^2} + \frac{(ad-bc)^2}{(c^2+d^2)^2},$$

and

$$\frac{A}{B} = \frac{(a^2+b^2)(c^2+d^2)}{(c^2+d^2)^2} = \frac{(ac-bd)^2+(ad+bc)^2}{(c^2+d^2)^2} = \frac{(ac-bd)^2}{(c^2+d^2)^2} + \frac{(ad+bc)^2}{(c^2+d^2)^2},$$

in accordance with Propositions XII. and VII.

XIII. No number comprising the sum of two squares, multiplied by another which is not the sum of two squares, will give a product consisting of the sum of two squares.

Proof. — Let P be a number which is not the sum of two squares; then $P(c^2+d^2)$ will not be. For, if it be possible, let

$$P(c^2+d^2) = a^2+b^2.$$

Then, dividing, we have, by the last proposition, $P = \frac{a^2+b^2}{c^2+d^2}$ = the sum of two squares, which is absurd.

XIV. No number which is the sum of two squares, divided by another not the sum of two squares, will give a quotient consisting of the sum of two squares.

Proof. — Let $\frac{a^2+b^2}{P} = c^2+d^2$; then $a^2+b^2 = P(c^2+d^2)$, which has just been shown to be impossible.

XV. When P and Q are prime to each other, and neither of them is the sum of two squares, then their product, $P \,.\, Q$, cannot be the sum of two squares.

Proof. — Let $Q = x+y$, then $P \,.\, Q = Px + Py$. Now Px can be a square only when $x = m^2 P^{2r+1}$, and Py can be a square only when $y = n^2 P^{2r+1}$. But in this case we shall have

$$Q = x+y = m^2 P^{2r+1} + n^2 P^{2r+1},$$

an expression which is divisible by P; and therefore P and Q are not prime to each other, which is contrary to the supposition.

XVI. When P and Q have a common factor, and $P \,.\, Q$ is the sum

of two squares, their product will contain, first, the square of that factor, and, secondly, another factor which is the sum of two squares.

Proof. — Let $P = m P'$ and $Q = m Q'$; then $P Q = m^2 P' Q'$. Now let $P' Q' = a^2 + b^2$; $m^2 P' Q' = m^2 a^2 + m^2 b^2 =$ the sum of two squares. Again, suppose that $P' Q'$ is not the sum of two squares, and let $P' Q' = a^2 + n$; then $m^2 P' Q' = m^2 a^2 + m^2 n$, which is not the sum of two squares.

Cor. — As the expression $4x + 3$ does not contain the sum of two squares (Prop. II. Cor. 3), and therefore cannot divide the quantity $a^2 + b^2$ (Prop. XIV.), it follows that, if it be contained in the product $P Q$ at all, it can only be as the factor represented by m^2. Consequently, every number consisting of the sum of two squares that are prime to each other must itself be a prime, or the product of prime factors, each of which is the sum of two squares.

And hence no number can be the sum of two squares if it is divisible by another which is neither a square nor the sum of two squares, and if it contains, as a factor, no even power of the divisor.

XVII. The expression $4x + 1$ may always be represented as the product of the following prime factors, in which any of the variables may become 0, viz.: —

1. $(4x' + 1)(4x'' + 1)(4x''' + 1)$, &c., indefinitely.

2. $(4x^{IV} + 3)(4x^{V} + 1)$ &c., indefinitely, the number of factors being even.

3. The product of the two preceding.

Proof. — If the multiplication, as indicated, be actually performed, the respective products will be found to be of the same form as the original expression, that is, $4x + 1$.

XVIII. All prime numbers of the form $4x + 1$ consist of the sum of two squares.

Proof. — 1. The sum of two even squares is even, and therefore cannot be prime, or of the form $4x + 1$. The same is true of two odd squares.

2. No number which is the sum of an odd and an even square is contained in the expression $(4x^{IV}+3)(4x^{V}+3)$, &c. (Prop. II. Cor. 3, and Prop. XIII.)

3. The product $(4x'+1)(4x''+1)(4x'''+1)$, &c. $\times(4x^{IV}+3)(4x^{V}+3)$, &c., contains no factor which is not to be found in the expressions 1 and 2 of the preceding proposition. Therefore,

4. All numbers containing the sum of an odd and an even square that are prime to each other, must be comprised in the expression $(4x'+1)(4x''+1)(4x'''+1)$ &c. But if there is any one of those factors, however large the number, which is not the sum of two squares, then, by Prop. XVI. and the corollary, it is manifest that the product cannot be; and thus we arrive at the absurd conclusion, that no number whatever can be the sum of an odd and of an even square. The truth of the proposition is therefore established.

XIX. No number that contains the sum of two squares in two different ways can be a prime number.

Proof. — Let $N=m^2+p^2=n^2+q^2$; then we have

$$m^2-n^2=q^2-p^2, \quad \text{or} \quad m+n:q+p :: q-p:m-n.$$

Again, let us put

$$m=ac+bd, \qquad p=ad-bc,$$
$$n=ac-bd, \qquad q=ad+bc.$$

Then
$$m+n=2ac, \qquad q+p=2ad,$$
$$m-n=2bd, \qquad q-p=2bc.$$

Whence we have the proportions,

$$m+n:q+p :: qac:2ad :: c:d,$$

and
$$q+p:m-n :: 2ad:2bd :: a:b.$$

Now $m+n$, $q+p$, $m-n$, are manifestly integers, since m, n, q, p are integers (Ind. Anal.); and reducing them to their lowest terms, we obtain integral values for a, b, c, d. (Prop. IX.)

Substituting now, for m, n, p, q, their equivalents, $ac+bd$, $ac-bd$, $ad-bc$, $ad+bc$, and reversing the process in Prop. X., we have $N=m^2+p^2=n^2+q^2=(ac\pm bd)^2+(ad\mp bc)^2=(a^2+b^2)(c^2+d^2)=$ the product of two integral factors. Therefore N is not prime.

Cor. — Therefore no prime number of the form $4x+1$ can be the sum of two squares in more than one way.

XX. Let there be an indefinite number of prime factors of the form $4x+1$, as $(4x'+1),(4x''+1),(4x'''+1)$, &c.; and let n represent the number of these factors contained in any given number, N. Then will 2^{n-1} denote the number of different ways in which N may be resolved into the sum of two squares in the manner already indicated.

Proof. — Since $4x'+1=a^2+b^2$, and $4x''+1=c^2+d^2$, their product $(4x'+1)(4x''+1)=(a^2+b^2)(c^2+d^2)=$ the sum of two squares in two different ways, according to Prop. X.

Let us represent these sets of squares by $m^2+n^2=p^2+q^2$. Multiplying now by $4x'''+1=e^2+f^2$, we have

$$(4x'+1)(4x''+1)(4x'''+1)=(a^2+b^2)(c^2+d^2)(e^2+f^2)=$$
$$(m^2+n^2)(e^2+f^2)=r^2+s^2=t^2+u^2,$$
and
$$(p^2+q^2)(e^2+f^2)=v^2+w^2=x^2+y^2.$$

Thus the product of three of those factors may be resolved into the sum of two squares in four different ways. If we introduce another factor, as $4x^{iv}+1=g^2+h^2$, the product

$$(4x'+1)(4x''+1)(4x'''+1)(4x^{iv}+1)=(a^2+b^2)(c^2+d^2)(e^2+f^2)(g^2+h^2)$$

would give us, in the same manner, the sum of two squares in eight different ways. Introducing another factor still, and proceeding as before, we should obtain a number which could be resolved into two squares in sixteen different ways; and so on as far as we please.

Thus we have a geometrical progression whose first term is 1, ratio 2, and the number of terms n. The nth term is therefore 2^{n-1}.

Remark. — This, however, does not give the whole number of resolutions of which a number may be susceptible, but only those of a particular form. Other forms may be obtained by the simple multiplication of expressions denoting squares. Thus, if $(a^2+b^2)=N^2$, then not only does $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$ (Prop. X. 4), but we have also

$$(a^2+b^2)^2=(a^2+b^2)(a^2+b^2)=a^2(a^2+b^2)+b^2(a^2+b^2)=a^2N^2+b^2N^2.$$

It is only by combining both forms that the whole number of solutions can be found.

XXI. If a number consist of the sum of three squares, the square of that number will consist of three squares in three different ways.

Proof. — Let $a^2+b^2+c^2$ be the number; then

$$\begin{aligned}(a^2+b^2+c^2)^2 &= (a^2-b^2-c^2)^2+4a^2b^2+4a^2c^2,\\ &= (-a^2+b^2-c^2)^2+4a^2b^2+4b^2c^2,\\ &= (-a^2-b^2+c^2)^2+4a^2c^2+4b^2c^2,\end{aligned}$$

as may be shown by the development.

XXII. In general, if a number is the sum of n squares, the square of that number will also be the sum of n squares in n different ways.

Proof. — Let $a^2+b^2+c^2+d^2+$ &c. to n terms be the number; then

$$\begin{aligned}&(a^2+b^2+c^2+d^2+\text{\&c. to } n \text{ terms})^2\\ &\quad=(a^2-b^2-c^2-d^2-\text{\&c. to } n \text{ terms})^2+4a^2b^2+4a^2c^2+4a^2d^2+\text{\&c. to } n \text{ terms},\end{aligned}$$

as may be shown in the same manner; and the other forms may be exhibited by merely changing the signs as above.

Remark. — This includes, of course, the case of two squares, or the formula $(a^2+b^2)^2=(a^2-b^2)+4a^2b^2$, as found by Proposition X. It is therefore of the utmost generality.

Problems.

1. To find two square numbers whose sum shall be a square.

Formula. $(a^2+b^2)^2=(a^2-b^2)+4a^2b^2$. (Prop. X. 4.)

Rule. — Take any two square numbers whatever, and add, subtract, and multiply them. Then the sum and difference will be the roots of two of the squares, and four times the product will be the other square.

Another Formula.

$$\{x(x+2a)+2a^2\}^2=\{x(x+2a)\}^2+\{ax+(ax+2a^2)\}^2;$$

or, making $x=1$,

$$\{x(x+2)+2\}^2=\{x(x+2)\}^2+\{x+(x+2)\}^2.$$

Rule. — Take any two numbers whose difference is 2. Their sum will be the root of one square, their product that of another, and 2 added to the product will give the root of the third.

This is merely a particular form of the preceding, and is deducible from it.

2. To find two square numbers whose difference shall be a square.

Formula. $(a^2-b^2)^2=(a^2+b^2)-4a^2b^2$ (Prop. XI., (10)),

in which a and b may be taken at pleasure.

This is the same with the one above, one of the terms being transferred to the other member.

3. To find two squares having a given difference.

Let $a^2-b^2=ff'$, f and f' being any two factors of the given difference whatever. Put $a+b=f$, and $a-b=f'$; then, adding, subtracting, and dividing by 2, we obtain $a=\frac{f+f'}{2}$, and $b=\frac{f-f'}{2}$, and these values of a and b will satisfy the conditions; for we shall have

$$(a+b)(a-b)=a^2-b^2=\left(\frac{f+f'}{2}+\frac{f-f'}{2}\right)\left(\frac{f+f'}{2}-\frac{f-f'}{2}\right)=ff'.$$

Remark. — If the difference be an odd number, its factors must be odd, and consequently their sum or difference will be even. In any

odd number, one factor may be unity, and the other the number itself. Thus every odd number is the difference of two integral squares.

4. To find two squares whose sum shall be unity.

Divide both members of the equation $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$ by $(a^2+b^2)^2$, and we shall have $1=\frac{(a^2-b^2)^2}{(a^2+b^2)^2}+\frac{4a^2b^2}{(a^2+b^2)^2}$; and the problem is solved.

5. To find three squares whose sum shall be unity.

Divide both members of the equation

$$(a^2+b^2+c^2)^2=(a^2-b^2-c^2)^2+4a^2b^2+a^2c^2 \text{ (Prop. XXI.)}$$

by $(a^2+b^2+c^2)^2$, and we obtain

$$1=\frac{(a^2-b^2-c^2)^2}{(a^2+b^2+c^2)^2}+\frac{4a^2b^2}{(a^2+b^2+c^2)^2}+\frac{4a^2c^2}{(a^2+b^2+c^2)^2},$$

which gives us the numbers required.

In the same way we may find any number of squares whose sum shall be unity; and by using the different equations in Propositions XXI., XXII., we may find as many different sets of square numbers whose sum shall be unity.

6. To find any number of squares whose sum shall be a square.

Let these squares be represented by the several expressions, $(a^2-b^2-c^2-d^2-$ &c. to n terms$)^2$, $4a^2b^2$, $4a^2c^2$, $4a^2d^2$, &c., to the required number n, and whatever be the values of a, b, c, &c., the problem will be solved. (Prop. XXII.)

7. To resolve any given square number, as N^2, into the sum of two squares.

Multiply the equation $1=\frac{(a^2-b^2)^2}{(a^2+b^2)^2}+\frac{4a^2b^2}{(a^2+b^2)^2}$, in Problem 4, by N^2, and $N^2=\frac{(a^2-b^2)^2}{(a^2+b^2)^2}\cdot N^2+\frac{4a^2b^2}{(a^2+b^2)}\cdot N^2$ is the resolution sought.

8. To resolve any given square number, as N^2, into the sum of three squares.

Multiply the equation

$$1 = \frac{(a^2-b^2-c^2)^2}{(a^2+b^2+c^2)^2} + \frac{4a^2b^2}{(a^2+b^2+c^2)^2} + \frac{4a^2c^2}{(a^2+b^2+c^2)^2},$$

in Problem 5, by N^2, and we shall have

$$N^2 = \frac{(a^2-b^2-c^2)^2}{(a^2+b^2+c^2)^2}\cdot N^2 + \frac{4a^2b^2}{(a^2+b^2+c^2)^2}\cdot N^2 + \frac{4a^2c^2}{(a^2+b^2+c^2)^2}\cdot N^2;$$

and the work is accomplished.

This process may be extended to any number of squares.

9. To resolve a number consisting of the sum of two squares into two other squares, in two different ways.

First find $c^2 + d^2 = 1$, by Prob. 4. Then we have, by Prop. X. 8,

$$a^2 + b^2 = (ac + bd)^2 + (ad - bc)^2,$$

and

$$a^2 + b^2 = (ac - bd)^2 + (ad + bc)^2,$$

by which the problem is solved.

10. To find a square number which is the sum of two squares in two different ways.

$(a^2 + b^2)^2 = (a^2 - b^2)^2 + 4a^2b^2$, as already proved. Squaring the equation,

$$(a^2+b^2)^4 = \{(a^2-b^2)^2+4a^2b^2\}^2 = \{(a^2-b^2)^2-4a^2b^2\}^2+16a^2b^2(a^2-b^2)^2.$$

Again, $(a^2+b^2)^4 = (a^2+b^2)^2(a^2+b^2)^2 = (a^2+b^2)^2(a^2-b^2)^2+4a^2b^2(a^2+b^2)^2$.

Thus $(a^2 + b^2)^4$ is the number sought.

11. To find a square number consisting of the sum of two squares, in twelve different ways.

By Prob. 10,

$$(a^2+b^2)^4 = \{(a^2-b^2)^2-4a^2b^2\}^2+16a^2b^2(a^2-b^2)^2 = (a^4-b^4)^2+4a^2b^2(a^2+b^2)^2.$$

Therefore,

$$(c^2+d^2)^4 = \{(c^2-d^2)^2-4c^2d^2\}^2+16c^2d^2(c^2-d^2)^2 = (c^4-d^4)^2+4c^2d^2(c^2+d^2)^2.$$

Substituting for these values $(a^2 + b^2)^4 = m^2 + n^2 = m'^2 + n'^2$, and $(b^2 + d^2)^4 = r^2 + s^2 = r'^2 + s'^2$, we have

$$(a^2+b^2)^4(c^2+d^2)^4 = (m^2+n^2)(r^2+s^2) = (mr \pm ns)^2 + (ms \mp nr)^2, \quad 2 \text{ ways},$$
$$(m^2+n^2)(r'^2+s'^2) = (mr' \pm ns')^2 + (ms' \mp nr')^2, \quad \text{“}$$
$$(m'^2+n'^2)(r^2+s^2) = (m'r \pm n's)^2 + (m's \mp n'r)^2, \quad \text{“}$$
$$(m'^2+n'^2)(r'^2+s'^2) = (m'r' \pm n's')^2 + (m's' \mp n'r')^2, \quad \text{“}$$

making, thus far, eight sets of squares, according to Prop. XX. But

$$(m^2+n^2)(r^2+s^2) = r^2(a^2+b^2)^4 + s^2(a^2+b^2)^4,$$
$$(m^2+n^2)(r^2+s^2) = m^2(a^2+b^2)^4 + n^2(a^2+b^2)^4,$$
$$(m'^2+n'^2)(r'^2+s'^2) = r'^2(a^2+b^2)^4 + s'^2(a^2+b^2)^4,$$
$$(m'^2+n'^2)(r'^2+s'^2) = m'^2(a^2+b^2)^4 + n'^2(a^2+b^2)^4;$$

four squares more obtained by multiplication, which complete the number required.

Restoring now to $m, n, m', n', r, s, r', s'$ their respective values, we obtain the formulæ

$$\begin{aligned}
&(a^2+b^2)^4(c^2+d^2)^4 \\
&= \{[(a^2-b^2)^2 - 4a^2b^2][(c^2-d^2)^2 - 4c^2d^2] \pm 16abcd(a^2-b^2)(c^2-d^2)\}^2 \\
&\qquad + 4cd(c^2-d^2)\{(a^2-b^2)^2 - 4a^2b^2\} \mp 4ab(a^2-b^2)\{(c^2-d^2) - 4c^2d^2\}^2 \\
&= \{[(c^2-d^2)^2 - 4c^2d^2](a^4-b^4) \pm 8abcd(c^2-d^2)(a^2+b^2)\}^2 \\
&\qquad + 2ab(a^2+b^2)\{(c^2-d^2) - 4c^2d^2\} \mp 4cd(c^2-d^2)(a^4-b^4)\}^2 \\
&= \{[(a^2-b^2)^2 - 4a^2b^2](c^4-d^4) \pm 8abcd(a^2-b^2)(c^2+d^2)\}^2 \\
&\qquad + 2dc(c^2+d^2)\{(a^2-b^2) - 4a^2b^2\} \mp 4ab(a^2-b^2)(c^4-d^4)\}^2 \\
&= \{(a^4-b^4)(c^4-d^4) \pm 4abcd(a^2+b^2)(c^2+d^2)\}^2 \\
&\qquad + \{2dc(c^2+d^2)(a^4-b^4) \mp 2ab(a^2+b^2)(c^4-d^4)\}^2 \\
&= (a^2+b^2)^4(c^4-d^4) + (a^2+b^2)^4\{4c^2d^2(c^2+d)^2\}^2 \\
&= (a^2+b^2)^4\{(c^2-d^2)^2 - 4c^2d^2\}^2 + (a^2+b^2)^4 . 16c^2d^2(c^2-d^2)^2 \\
&= (c^2+d^2)^4(a^4-b^4)^2 + (c^2+d^2)^4 . 4a^2b^2(a^2+b^2)^2 \\
&= (c^2+d^2)^4\{(a^2-b^2)^2 - 4a^2b^2\}^2 + (c^2+d^2)^4 . 16a^2b^2(a^2-b^2)^2.
\end{aligned}$$

Let $a = 2$, $b = 1$, $c = 3$, $d = 2$, we shall then have

$$(a^2+b^2)^4(c^2+d^2)^4 = 65^4 = 4225^2$$

$$\begin{aligned}
&= 2047^2 + 3696^2, && = 2145^2 + 3640^2, \\
&= 3713^2 + 2016^2, && = 4095^2 + 1040^2, \\
&= \ 615^2 + 4180^2, && = 2535^2 + 3380^2, \\
&= 4185^2 + \ 580^2, && = 1183^2 + 4056^2, \\
&= 3289^2 + 2652^2, && = 1625^2 + 3900^2, \\
&= 4199^2 + \ 468^2, && = 2975^2 + 3000^2.
\end{aligned}$$

The number 65^5 may be resolved into the sum of two squares, according to this method, in eighty-four different ways.

12. To find a cube equal to the sum of two squares.

Multiply the identical equation $(a^2+b^2)=a^2+b^2$ by $(a^2+b^2)^2$, and we shall have

$$(a^2+b^2)^3=a^2(a^2+b^2)^2+b^2(a^2+b^2)^2; \qquad \text{(A)}$$

otherwise, $$(a^2+b^2)^3=a^2(a^2-3b^2)^2+b^2(3a^2+b^2)^2. \qquad \text{(B)}$$

Thus the problem may be solved in two different ways.

Remark. — Since $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$, by substituting these values in equation (A), we obtain

$$(a^2+b^2)^3=a^2(a^2-b^2)^2+4a^4b^2+b^2(a^2-b^2)^2+4a^2b^4.$$

Thus we may find four squares whose sum shall be a cube, or we may find a cube which is capable of being resolved into four squares.

13. To find a biquadrate composed of two squares.

Multiply the equation $(a^2+b^2)^2=(a^2-b^2)^2+4a^2b^2$, by $(a^2+b^2)^2$, and we have $(a^2+b^2)^4=(a^4-b^4)^2+4a^2b^2(a^2+b^2)^2$, the required solution. Resolving $(a^2+b^2)^2$ in this last into its component squares, we get $(a^2+b^2)^4=(a^4-b^4)^2+4a^2b^2(a^2-b^2)^2+16a^4b^4$, 3 squares. But $a^4-b^4=(a^2-b^2)(a^2+b^2)$; and therefore resolving again, we have $(a^2+b^2)^4=(a^2-b^2)^4+4a^2b^2(a^2-b^2)^2+4a^2b^2(a^2-b^2)^2+16a^4b^4$, 4 squares. Thus we find the same biquadrate equal to the sum of 2, 3, or 4 squares. If $a=2$ and $b=1$, we have

$$5^4=25^2=15^2+20^2=12^2+15^2+16^2=9^2+12^2+12^2+16^2.$$

14. To find a number whose fifth power shall be the sum of two squares in two ways.

Multiply the two formulæ in Prob. 12 by $(a^2+b^2)^2$, and we obtain

$$(a^2+b^2)^5=a^2(a^2+b^2)^4+b^2(a^2+b^2)^4,$$

and $$(a^2+b^2)^5=a^2(a^2+b^2)^2(a^2-3b^2)^2+b^2(a^2+b^2)(3a^2-b^2)^2.$$

$$(a^2+b^2)^4(c^2+d^2)^4 = (m^2+n^2)(r^2+s^2) = (mr \pm ns)^2 + (ms \mp nr)^2, \quad 2 \text{ ways},$$
$$(m^2+n^2)(r'^2+s'^2) = (mr' \pm ns')^2 + (ms' \mp nr')^2, \quad \text{“}$$
$$(m'^2+n'^2)(r^2+s^2) = (m'r \pm n's)^2 + (m's \mp n'r)^2, \quad \text{“}$$
$$(m'^2+n'^2)(r'^2+s'^2) = (m'r' \pm n's')^2 + (m's' \mp n'r')^2, \quad \text{“}$$

making, thus far, eight sets of squares, according to Prop. XX. But

$$(m^2+n^2)(r^2+s^2) = r^2(a^2+b^2)^4 + s^2(a^2+b^2)^4,$$
$$(m^2+n^2)(r^2+s^2) = m^2(a^2+b^2)^4 + n^2(a^2+b^2)^4,$$
$$(m'^2+n'^2)(r'^2+s'^2) = r'^2(a^2+b^2)^4 + s'^2(a^2+b^2)^4,$$
$$(m'^2+n'^2)(r'^2+s'^2) = m'^2(a^2+b^2)^4 + n'^2(a^2+b^2)^4;$$

four squares more obtained by multiplication, which complete the number required.

Restoring now to $m, n, m', n', r, s, r', s'$ their respective values, we obtain the formulæ

$$\begin{aligned}
&(a^2+b^2)^4(c^2+d^2)^4 \\
&= \{[(a^2-b^2)^2 - 4a^2b^2][(c^2-d^2)^2 - 4c^2d^2] \pm 16abcd(a^2-b^2)(c^2-d^2)\}^2 \\
&\qquad + 4cd(c^2-d^2)\{(a^2-b^2)^2 - 4a^2b^2\} \mp 4ab(a^2-b^2)\{(c^2-d^2) - 4c^2d^2\}^2 \\
&= \{[(c^2-d^2)^2 - 4c^2d^2](a^4-b^4) \pm 8abcd(c^2-d^2)(a^2+b^2)\}^2 \\
&\qquad + 2ab(a^2+b^2)\{(c^2-d^2) - 4c^2d^2\} \mp 4cd(c^2-d^2)(a^4-b^4)\}^2 \\
&= \{[(a^2-b^2)^2 - 4a^2b^2](c^4-d^4) \pm 8abcd(a^2-b^2)(c^2+d^2)\}^2 \\
&\qquad + 2dc(c^2+d^2)\{(a^2-b^2) - 4a^2b^2\} \mp 4ab(a^2-b^2)(c^4-d^4)\}^2 \\
&= \{(a^4-b^4)(c^4-d^4) \pm 4abcd(a^2+b^2)(c^2+d^2)\}^2 \\
&\qquad + \{2dc(c^2+d^2)(a^4-b^4) \mp 2ab(a^2+b^2)(c^4-d^4)\}^2 \\
&= (a^2+b^2)^4(c^4-d^4) + (a^2+b^2)^4\{4c^2d^2(c^2+d)^2\}^2 \\
&= (a^2+b^2)^4\{(c^2-d^2)^2 - 4c^2d^2\}^2 + (a^2+b^2)^4 . 16c^2d^2(c^2-d^2)^2 \\
&= (c^2+d^2)^4(a^4-b^4)^2 + (c^2+d^2)^4 . 4a^2b^2(a^2+b^2)^2 \\
&= (c^2+d^2)^4\{(a^2-b^2)^2 - 4a^2b^2\}^2 + (c^2+d^2)^4 . 16a^2b^2(a^2-b^2)^2.
\end{aligned}$$

Let $a = 2$, $b = 1$, $c = 3$, $d = 2$, we shall then have

$$(a^2+b^2)^4(c^2+d^2)^4 = 65^4 = 4225^2$$

$$\begin{aligned}
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&= 3289^2 + 2652^2, && = 1625^2 + 3900^2, \\
&= 4193^2 + 468^2, && = 2975^2 + 3000^2.
\end{aligned}$$

:t the sum of
ıole, shall be

$8a^2 - 8a + 4.$

$-4a+2)^2,$
$-2a+2)^2,$

$a^4 - 26a^2 + 25.$

$z = (a^2 - 7)^2,$

l y that $x^2 + y^2$ and

square when

(Prop. X. (3));

$x^2 = (a^2 + b^2)^2$ and
$-b^2)^2$ in one case, and
ame; and if $y^2 = 4a^2b^2$
value is not the same.

both be squares at the
$-y^2) = x^4 - y^4 = z^2$, is
impossible, and, finally,

A larger number of squares may be found in this, as in the preceding problems, by resolving $(a^2+b^2)^2$ into the squares which compose it.

15. To find a number whose sixth power is the sum of two squares.

Multiply the formula for Prob. 13 by $(a^2+b^2)^2$, and we get

$$(a^2+b^2)^6 = (a^2+b^2)^2(a^4-b^4)^2 + 4a^2b^2(a^2+b^2)^4.$$

This process may be continued indefinitely, and thus we may find any number of squares whose sum shall be of any proposed power.

16. To assign such values to x and y, that x^2+y and x^2-y may both be squares.

Let $x = a^2+b^2$, and $y = 4ab(a^2-b^2)$; then

$$x^2+y=(a^2+b^2)^2+4ab(a^2-b^2)=(a^2-b^2)^2+4ab(a^2-b^2)+4a^2b^2=\{(a^2-b^2)+2ab\}^2,$$

and

$$x^2-y=(a^2+b^2)^2-4ab(a^2-b^2)=(a^2-b^2)^2-4ab(a^2-b^2)+4a^2b^2=\{(a^2-b^2)-2ab\}^2.$$

Therefore these values will satisfy the conditions.

17. To assign such values to x and y, that $x^2 \pm cy$ may be a square.

Let $x = c(a^2+b^2)$, and $y = c \cdot 4ab(a^2-b^2)$. Then we have $x^2 \pm cy = c^2(a^2+b^2)^2 \pm c^2 \cdot 4ab(a^2-b^2)$; which is the same as the preceding multiplied by c^2.

18. To assign such a value to x that $x^2 \pm cx$ shall be a square.

Let $x^2 = (a^2+b^2)^2$, and $x = \frac{4ab(a^2-b^2)}{c}$. Combining these equations by dividing the former by the latter, so that the value of x may be the same in both, we have $x = \frac{c^2(a^2+b^2)^2}{4ab(a^2-b^2)}$. Then

$$x^2 \pm cx = \frac{c^2(a^2+b^2)^4}{16a^2b^2(a^2-b^2)^2} \pm \frac{c^2(a^2+b^2)^2}{4ab(a^2-b^2)}.$$

Reducing to a common denominator, and then rejecting the denominator, which is a square, and simplifying the expression, we obtain $x^2 \pm cx = c^2(a^2+b^2)^2\{(a^2-b^2) \pm 2ab\}^2$, a square.

19. To assign such values to the numbers x, y, z, that the sum of every two of them, and likewise the sum of the whole, shall be squares.

Let $x = 4ab^2 + 4b^3 - 12b^2 + 8b$,

$y = -4ab^2 - 4b^3 + 12b^2 - 8b + 4a^2b^2$,

$z = -4ab^2 + b^4 + 4b^2 + a^4 - 2a^2b^2 - 4a^2 + 8a^2 - 8a + 4$.

Then

$x + y = (2ab)^2$, $\quad y + z = (b^2 + a^2 - 4a + 2)^2$,

$x + z = (b^2 - a^2 + 2)^2$, $\quad x + y + z = (b^2 + a^2 - 2a + 2)^2$,

and all the conditions are satisfied.

Otherwise, let

$x = 24a^2 - 24$, $\quad y = 12a^2 + 24$, $\quad z = a^4 - 26a^2 + 25$.

Then $x + y = 36a^2$, $\quad x + z = (a^2 - 1)^2$, $\quad y + z = (a^2 - 7)^2$,

and $x + y + z = (a^2 + 5)^2$.

The former method is the more general.

20. It is required to find such values for x and y that $x^2 + y^2$ and $x^2 - y^2$ shall both be squares.

This is impossible; for $x^2 + y^2$ can only be a square when

$x^2 = (a^2 - b^2)^2$ and $y^2 = 4a^2b^2$ (Prop. X. (3));

and $x^2 - y^2$ can only be a square when $x^2 = (a^2 + b^2)^2$ and $y^2 = 4a^2b^2$, or $(a^2 - b^2)^2$. But if $x^2 = (a^2 - b^2)^2$ in one case, and $(a^2 + b^2)^2$ in the other, the value is not the same; and if $y^2 = 4a^2b^2$ in one case, and $(a^2 - b^2)^2$ in the other, the value is not the same. Therefore the problem cannot be solved.

Further, if $x^2 + y^2$ and $x^2 - y^2$ cannot both be squares at the same time, then their product, $(x^2 + y^2)(x^2 - y^2) = x^4 - y^4 = z^2$, is impossible. Therefore $x^4 - y^4 = z^4$ is impossible, and, finally, $x^4 = y^4 + z^4$ is impossible.

21. To ascertain whether any given number is the sum of two squares in one or more ways, and if so, to find them.

1. Divide the number by 4, and if there be a remainder of 3 it cannot be the sum of any two squares whatever. (Prop. II. Cor. 3.)

2. See whether it can be divided by any number of the series contained in the expression $4x + 3$, as 3, 7, 11, &c. If it can, and not by any even power of that number, then also it cannot be the sum of two squares. (Prop. XVI. Cor.)

3. In all other cases the number will consist of the sum of two squares in one or more ways.

Separate the number into its square factors, if it have any, and also into its prime factors. Each of the latter will consist of the sum of two squares, which may easily be found by trial (Prop. XVIII.). Find them, and resolve their product, as in Prop. X.

Thus, let $N = m^2(a^2 + b^2)(c^2 + d^2)$. Having found, as directed, the several factors, and then the values of a, b, c, d, we have, by Prop. X., $(a^2 + b^2)(c^2 + d^2) = (ac \pm bd)^2 + (ad \mp bc)^2$ = the sum of two squares, in two different ways. Now, multiplying by m^2, we have $m^2(a^2 + b^2)(c^2 + d^2) = m^2(ac \pm bd)^2 + m^2(ad \mp bc)^2$, the resolution required.

Thus, 585 divided by 4 gives the remainder 1; it may, therefore, as far as we have yet learned, be the sum of two squares. Resolve it into its several prime factors. They are $3^2 . 5 . 13$. $5 = 2^2 + 1^2$, and $13 = 3^2 + 2^2$. Wherefore,

$$(2^2 + 1^2)(3^2 + 2^2) = (6 \pm 2)^2 + (4 \mp 3)^2 = 64 + 1 = 16 + 49 = 65.$$

Multiplying these results by $3^2 = 9$, we have

$$9 . 65 = 585 = 576 + 9 = 144 + 441,$$

the squares which were sought.

NOTES ON THE THEORY OF PROBABILITY.—INSURANCE.

By Simon Newcomb, Nautical Almanac Office, Cambridge, Mass.

In the first volume of the Mathematical Monthly it was remarked, that the value of a probability of obtaining any sum of money was equal to the product of the sum into the fraction expressing the probability. But this was only on the hypothesis that the desirableness of the money was the same, whether it was all in the hands of one man, or divided among several. This hypothesis is by no means universally true; hence the real value of the expectation of receiving money cannot always be estimated by this rule.

The value or desirableness of a small sum, say one dollar, is much greater to a man who is poor than to one who is wealthy; and as the same man grows richer, the value of one dollar to him will grow less and less. The value of money is to be estimated by the desirableness of the commodities which are purchased with it. A man will first spend his money for articles which are quite indispensable. As he is able to spend more, he spends it in articles which are less and less desirable. If, now, we had for any man some continuous function which should express a relationship between the desirableness and the money value of commodities in general, we should be able to express the value of any sum in terms of its nominal amount. It is obviously impossible in any case to obtain such a function by any rigorous process. But an hypothesis, near enough for practical purposes, has been framed. This hypothesis is, that the value of one dollar to any man is inversely proportional to his wealth; or, perhaps better still, inversely proportional to his annual income, or to the amount of money which he can afford to spend in a year.

If, independently of his income, he possesses productive property, it is perhaps fair to presume that the value of a dollar to him will be

the same as if his annual income were increased by one tenth of the amount of the property. On all these hypotheses, if we represent by v the moral value, or desirableness, of a unit of money, say one dollar, and by s the absolute amount of money, reckoned by dollars, and by i the annual income, we shall have

$$dv = \frac{i\,ds}{i + \frac{1}{10}s}. \tag{1}$$

The unit of absolute value is here taken to be the value of one dollar when the individual has nothing but his annual income. s being then zero, we have $dv = ds$. As s increases, the value of $\frac{dv}{ds}$ will diminish in the same proportion that $i + \frac{1}{10}s$ increases, as is required by hypothesis. Integrating (1), we have

$$v = \log c(s + 10i)^{10i},$$

c being an arbitrary constant. Taking this so that v shall represent the moral value of s, we have

$$v = \log\left(1 + \frac{s}{10i}\right)^{10i}, \tag{2}$$

which, when s is small, gives, very nearly, $v = s$, as it should. But when s grows very large, v does not increase so rapidly as s. But if we make s negative, we find that the negative value of v will be much greater than that of s. Both of these results are perfectly in accordance with the judgment of common sense. If a man loses a considerable amount, his increased poverty increases the moral value of a given amount of money, and thus his moral loss exceeds the loss in dollars. If he gains a large sum, his increased wealth diminishes the moral value of money, and thus his moral gain is less than that in dollars.

From these facts arise the benefit of insurance and the evil of gambling. Everything which causes arbitrary transfers of money

from one person to another is necessarily an injury, even when the more remote evil effects are left out of consideration. From what has been seen, it will be perceived that the actual loss of the loser is *greater* than the value of the money he parts with, while the winning of the gainer is *less*. Thus, on the whole, there is a moral loss.

Again, everything which tends to prevent sudden and disastrous fluctuations of wealth is beneficial. Especially is this the case where an unforeseen loss can be equitably divided among a large number, instead of falling on a single person. This is the effect of insurance. Thus the difference between those contingent payments which are beneficial and those which are injurious admits of perfect mathematical definition. The class to which they belong depends on whether they are designed to prevent or to cause fluctuations in individual wealth.

Life Insurance.

The advantage of life insurance is founded on the fact, that to the heirs of a person who has no resource but an annual income, money will, after his death, be more valuable than during his life. He therefore lays up a certain sum annually. But it is entirely uncertain to what his savings may amount at his death, and in order to avoid this uncertainty, he compounds with the insurance company by paying it the amount of his annual saving which he wishes to devote to his family after his decease, on condition that they shall, on his decease, pay his family the probable accumulation of his earnings, be the actual accumulation great or small. To show clearly the manner in which this is effected, let us consider successively several possible forms of mutual life-insurance companies. We shall first proceed on the supposition that the actual experience coincides exactly with the probable law of mortality.

1. Out of 1,000 persons aged 25, about 8 die annually. That number of persons at that age may then form a company, and, desiring that each shall receive $1,000 on his death, agree to pay annually the product of $1,000, in the probability that he will die during the year. Each will therefore commence with the annual payment of eight dollars, which will just suffice to pay the policies of the eight deaths which will occur during each year. As the living ones grow older, a larger proportion of them will die; the premium must therefore be increased. At the age of forty it will amount to ten dollars, and at the age of eighty to about a hundred. In this form of company there would be no accumulation of funds, since the amount of premiums each year would only suffice to pay the losses. The policies at the ends of the several years would always be valueless; hence a member could withdraw at any time without suffering any loss, except what arose from his fortune in continuing to live. He would, in fact, have been insuring his life from year to year in the same manner that he would insure a house.

This form of company would have the disadvantage of obliging those who attained a considerable age either to give up their policies or pay enormous premiums. To avoid this, the earlier premiums are increased, and the premium made constant during life; its amount being so regulated that the accumulations of the earlier years with their interest shall just suffice to make up the deficiencies of the later ones. In the case which we have supposed, the constant annual payment, reckoning interest at four per cent, would be about fourteen dollars. The six thousand dollars extra per annum which would thus at first accumulate, would be put at interest for the benefit of those who should live to old age. As the members of the company died off, the annual receipts would be continually lessened, while at the same time, owing to the greater number of deaths, the disbursements would increase. Thus, in thirty years the payments

would exceed the income, the accumulated fund would be drawn upon, and finally, when the last man died, it would be exhausted in paying his policy.

The amount of the accumulated fund at any time shows the *value of the policies.* It is, indeed, the difference between the present value of all the future annual payments and the present value of the expectation of the value of the policy at the time of decease. To explain this mathematically, let $\varphi . t$ represent the probability that an individual will be alive at the end of t years. $\varphi . t$ must be equal to unity when t is zero, and vanish for $t = \infty$. It cannot be exactly expressed by any simple function, but is tabulated in tables of mortality. The present value of a premium, a, to be paid in t years, will be $\frac{a}{(1+r)^t}$, r being the rate of interest. But if the payment is to be made only on condition that an individual is living at that time, the present value of the expectation will be equal to the above multiplied by the probability that the man will then be alive, or by $\varphi . t$. It is therefore equal to $\frac{a . \varphi . t}{(1+r)^t}$. The present value of the expectation of a series of annual premiums, the first one being paid down, is

$$a \left\{ 1 + \frac{\varphi . 1}{1+r} + \frac{\varphi . 2}{(1+r)^2} + \frac{\varphi . 3}{(1+r)^3} + \text{\&c.} \right\} \qquad (3)$$

The probability that the individual will die between the times t and $t + dt$ is $d . \varphi t$ or $\varphi' . t\, dt$. The present value of the expectation of a sum, S, to be paid if he dies at that time, is $S \frac{\varphi' t\, dt}{(1+r)^t}$. The sum being payable whenever the man dies, the complete present value of the expectation is the integral of this expression, or

$$S \int_0^\infty \frac{\varphi' t}{(1+r)^t}\, dt. \qquad (4)$$

If the man is but entering the company, his annual premium is

determined by the condition that the two expressions above shall be equal, which gives

$$\frac{a}{S} = \int_0^\infty \frac{\varphi' t}{(1+r)^t}\, dt \div \sum_0^\infty{}_t \frac{\varphi . t}{(1+r)^t}. \qquad (5)$$

The values of these integrals are determined numerically from tables of mortality, by the method of mechanical integration.

The form of the function φ is such, that the numerical value of $\varphi' t$ becomes greater the older the man grows, whence a larger annual premium is requisite. Since, in the company which we have supposed, the expectation of final profit is nothing, the present value of the expectation of the policy payable on death must exceed the present value of the expectation of future premiums by that portion of the accumulation which belongs to the policy. This accumulation, therefore, represents the value of the policy.

This form of company requires a modification, from two causes. In the first place, it is probable that the actual rate of mortality will differ slightly from that of the tables, a contingency which must be provided against; and again, the actual rate of interest may differ from that used in calculating the premiums. In the next place, expenses must be provided for. Hence the theoretical premium is increased so as to provide for expenses and for possible depreciations in the rate of interest, and for possible extraordinary numbers of deaths. This increase is termed the loading, and usually amounts to about one fourth the amount of the theoretical premium.

Owing to the fact, that the actual rate of interest is generally much higher than that on which the premiums are calculated (the latter being usually but four per cent), it is almost certain that there will be a considerable accumulation in the treasury over and above the value of the policies. This surplus ought, at stated intervals, to be paid back to, or at least placed to the credit of, those policy-

holders from whose premiums it arose. If this is not done, or if the entire amount of this extra accumulation is not thus disposed of, it will be unfair to the earlier members, whose money will go to the benefit of future members; if more than the surplus is divided, it will be unfair to future policy-holders, who will ultimately be obliged to make up the deficiency. The amount which ought to be distributed is always susceptible of pretty exact mathematical calculations.

AWARD OF THE PRIZES FOR SOLUTIONS OF PROBLEMS IN No. VI., Vol. III.

The first prize is awarded to David Trowbridge, Perry City, N. Y.

The second prize is awarded to Asher B. Evans, Madison University, Hamilton, N. Y.

The third prize is awarded to W. L. Marcy, Coventry, N. Y.

The fourth prize is awarded to C. H. Bagley, Nunda, N. Y.

Prize Solution of Problem I.

By C. H. Bagley, Nunda Institute, Nunda, N. Y.

Given

(1) $a^3 - 3x^4 + (2+4y)x^3 + (4y+1-2a)x^2 - (2a+4ay)x = 4y^4$,

(2) $x = b + y$,

to find x and y by quadratics.

Adding $4x^2y^2 + 4x^4$ to both sides of (1), and extracting the square root, it becomes

$$a - x^2 - x - 2xy = \pm(y^2 + 2x^2).$$

Combining this with (2), we get

$$6x^2 + (1 - 4b)x = a - b^2, \quad \text{or} \quad x = a + b^2.$$

The values of x from these equations substituted in (2) will give the corresponding values of y.

Solutions of Prob. II. are essentially the same as the one found in most text-books on elementary geometry.

Prize Solution of Problem III.

By Asher B. Evans, Madison University, Hamilton, N. Y.

Which is greater, $2\tan^{-1}(\sqrt{-1})$, or $3\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{5}{99}$?

Put $2\theta = 2\tan^{-1}(\sqrt{2}-1)$, $3\varphi = 3\tan^{-1}\frac{1}{4}$, $\beta = \tan^{-1}\frac{5}{99}$; then $\tan\theta = \sqrt{2}-1$, $\tan\varphi = \frac{1}{4}$, $\tan\beta = \frac{5}{99}$. Employing the elementary formulas of trigonometry, we obtain

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta} = 1, \qquad \tan 2\varphi = \frac{2\tan\varphi}{1-\tan^2\varphi} = \frac{8}{15},$$

$$\tan 3\varphi = \frac{\tan 2\varphi + \tan\varphi}{1-\tan 2\varphi\tan\varphi} = \frac{47}{52}.$$

Hence, $$\tan(3\varphi+\beta) = \frac{\tan 3\varphi+\tan\beta}{1-\tan 3\varphi\tan\beta} = 1;$$

$$\therefore\ 3\varphi+\beta = \tan^{-1}(1) = 2\theta,$$

which shows that the two expressions are equal.

Prize Solution of Problem IV.

By David Trowbridge, Perry City, N. Y.

In any spherical triangle,

$$\frac{\sin s\sin(s-a)\sin(s-b)\sin(s-c)}{\cos S\cos(S-A)\cos(S-B)\cos(S-C)} = \frac{\cos a\cos b\cos c-1}{\cos A\cos B\cos C+1},$$

in which $s = \frac{1}{2}(a+b+c)$, $S = \frac{1}{2}(A+B+C)$.

Put the numerators of the first and second members equal to k^2 and n, and their denominators equal to K^2 and N. From trigonometry,

(1) $$\cos A = \frac{\cos a-\cos b\cos c}{\sin b\sin c}, \qquad (2)\quad \cos a = \frac{\cos A+\cos B\cos C}{\sin B\sin C}.$$

Multiply (1) by $\cos a$, and (2) by $\cos A$; then

(3) $$\cos a\cos A = \frac{\cos^2 a-\cos a\cos b\cos c}{\sin b\sin c} = \frac{n+\sin^2 a}{\sin b\sin c} = \frac{N-\sin A}{\sin B\sin C};$$

$$(4) \qquad \therefore \ \frac{n+\sin^2 a}{N-\sin^2 A} = -\frac{\sin b \sin c}{\sin B \sin C} = -\frac{\sin^2 a}{\sin^2 A} = \frac{n}{N}.$$

We also have

$$\sin A = \frac{2k}{\sin b \sin c}, \quad \sin a = \frac{2K\sqrt{-1}}{\sin B \sin C}; \quad \therefore \ \frac{k^2}{K^2} = -\frac{\sin^2 a}{\sin^2 A} = \frac{n}{N}.$$

Prize Solution of Problem V.

By David Trowbridge, Perry City, N. Y.

If the earth be completely covered by a sea of small depth, prove that the depth in latitude l is very nearly equal to $H(1-\varepsilon \sin^2 l)$, where H is the depth at the equator, and ε the ellipticity of the earth.

The depth will be inversely proportional to the force of gravity, which is very nearly inversely proportional to the square of the distance from the centre of the earth. Hence the depth will be nearly directly proportional to the square of the distance from the centre of the earth. If a is the distance from the centre at the equator, and r at the latitude l; then, if x is the depth at latitude l, we have

$$H : x = a^2 : r^2; \qquad \therefore \ x = \frac{Hr^2}{a^2}.$$

But $$r^2 = \frac{a^2 b^2}{a^2 \sin^2 l + b^2 \cos^2 l} = \frac{a^2(1-\varepsilon^2)}{1-\varepsilon^2+\varepsilon^2 \sin^2 l} = a^2(1-\varepsilon^2 \sin^2 l),$$

by developing according to the powers of ε, and retaining only ε^2.
$\therefore \ x = H(1-\varepsilon^2 \sin^2 l)$.

THE ÉCOLE DES PONTS ET CHAUSSÉES AT PARIS.

This school was founded by the celebrated engineer, Perronet, in 1746, for the purpose of instructing in a more thorough manner the engineers of the kingdom, who had previously been organized under the title *Corps des Ponts et Chaussées*. The functions of this corps are as follows: — 1st. To prepare the projects of the great public works undertaken by the state, such as river and harbor improvements, light-houses, roads, railroads, bridges, etc.; 2d. To superintend the construction of the works; 3d. To regulate the accounts and liquidate the expenses thus incurred.

Private companies executing similar enterprises have frequently intrusted their work, by an authorization of the government, to *Ingénieurs des Ponts et Chaussées*, these engineers being in effect temporarily detached from the service, without losing either their rank, or their right to advancement.

The instruction at the school is composed of two parts: — 1st. The instruction properly so called; 2d. The practical instruction of *Missions*.

The instruction proper consists of, — 1st. Lectures given by the Professors; 2d. Graphical Works, Executions of Projects, and Memoirs; 3d. Manipulations and Essays of Materials of Construction; 4th. Exercises in the Field of Levelling and Surveying; 5th. Visits to the Manufactories.

The studies at the school, including the examinations, last from the first of November until the first of June.

MISSIONS.

From the first of June to the first of November the students are sent into the Departments of France, and there attached to the public works which are in the course of execution, in order that they may be exercised, under the direction of the chiefs of the service, in the art of the engineer.

During the mission the students keep a journal of the instruction they gather, the observations they make, and the works in which they take part.

The students destined to recruit the *Corps des Ponts et Chaussées* are taken from among the graduates of the Polytechnic School.

In addition to these, the administration is authorized to receive some eight or ten students, French or foreigners, who, after a severe examination, are found qualified to enter. These are admitted to the same courses, make the same graphical works, submit to the same examinations, but are not entitled, when they graduate, to a position in the service of the state.

This examination comprises the subjects of Arithmetic, Algebra, Elementary Geometry, Trigonometry, Analytical Geometry, Descriptive Geometry, Shades, Shadows, and Perspective, Stonecutting, Carpentry, Differential and Integral Calculus, Analytic Mechanics, Cinematics, Architecture, Physics, Chemistry, and Design; and I can say from experience, that one cannot be in the school a month without needing to apply some principle of every one of the above-named subjects.

The following are the COURSES: —

1. *Course upon Routes.* — Including surveying, levelling, use and adjustment of instruments, earth-work, method of construction of roads, and the works accessory.

In connection with the course there are projects given. A topographical map, with two points upon the opposite borders, is given to the student, and he is required to trace a road between those points, calculate the excavations and embankments, the quantity of land necessary to purchase, cost of the transport of all the materials, cost of extraction of the pavage from the quarries, and, in fine, every possible item which can arise in the most complicated road, and afterwards to furnish plans for the bridges over the streams.

2. *Course of Resistance of Materials.* — Including applications of analytic mechanics to the various questions of the resistance of materials, methods for calculating the alteration in form of beams of iron or wood, and arcs of iron when loaded and subjected to variations of temperature, analytical investigation of the effects of vibration, thrust of roofs, pressure of earth, etc.

In this connection there are two projects given: — 1st. To calculate the strains which are produced in a straight iron bridge loaded; 2d. To calculate the strains, thrust, and variations of form of a bridge resting upon an iron arc, taking into account the variations of temperature.

3. *Course upon Hydraulics.* — Including analytical hydraulics and hydrodynamics; flowing of liquids from an orifice, in pipes, and in open canals; movement of gas; water-wheels, turbines, pumps.

Projects are given, — 1st. Upon the construction of water-wheels; 2d. Upon the distribution of water.

4. *Course upon Bridges.* — Comprising nature and properties of building-materials, their defects, means of preservation; practical details upon the execution of the work; bridges in masonry, in wood, iron bridges, suspension bridges.

A project for a grand bridge in masonry is given in connection with the course.

5. *Course upon Agricultural Hydraulics.* — Comprehending principles of agriculture; analysis of soils, nature and analysis of manures, agricultural meteorology, drainage, irrigations, etc.

There is a project given in this course for the amelioration of a piece of marsh land by drainage.

6. *Interior Navigation.* — Comprising processes for the amelioration of the navigation of rivers, dams, locks, canals; description of the principal canals of France; details of constructions, distribution of water, Artesian wells.

There are several projects given in connection with this course: — 1st. One for the construction of a dam; 2d. One for the construction of a *pont-canal*; 3d. A project to improve the river Yonne.

7. *Course upon Steam-Engines.* — Comprising theory of heat, history of steam-engines, properties of steam, calculation of the effect of steam-engines; stationary, locomotive, and marine engines; Ericsson's machines, special study of locomotives.

8. *Course upon Railroads.* — Comprising history of railroads, systems of rails in use, the details of the construction and execution of the work.

9. *Course upon Geology and Mineralogy.* — Comprising elements of crystallography, description of the minerals which enter into the composition of building materials, composition of the strata of the earth, formation of mountain-chains, volcanos and their eruptions, glaciers, etc.

10. *Course upon Harbors.* — Comprising a careful study of winds, tides, and currents; indications of the details for the improvement of harbors, building of docks, port-gates, etc.

A project is given for the construction of a grand dock.

11. *Architecture.* — Comprising composity of grand edifices, and details.

There are a great number of projects upon this subject, consisting of projects of a market, a city hall, hospital, railroad station, light-house, and a crystal palace.

12. *Course upon Political Economy;* and also one upon *Administrative Right.*

The students are taught the German and English languages. They are also exercised in sketching machines, making visits to the workshops for that purpose. They are exercised in sketching landscapes and making drawings in water-colors. There are attached to the school a large library, and an immense collection of models. A laboratory also, in which the students are exercised in the analysis of soils, essays of building-materials, and manipulations in photography.

The students are required to be present at the school from November to June daily, from the hours of 8¼ A. M. to 5 P. M., except one hour for breakfast; and during this time no one is allowed to leave the school without special permission from the officer of service. The rank of the students is determined by examinations, which take place once every six weeks, and also at the end of each year. The lectures delivered at the school, as well as most of the lectures delivered at Paris, are open to the public.

École des Ponts et Chaussées, Paris, 1861.

Mathematical Monthly Notices.

A History of the Progress of the Calculus of Variations during the Nineteenth Century. By I. TODHUNTER, M. A., Fellow and Principal Mathematical Lecturer of St. John's College, Cambridge. Macmillan & Co., Cambridge: and 23 Henrietta Street, Covent Garden, London. 1861.

We have looked for this volume with more than ordinary interest, believing that we greatly need at present works of this kind, covering all the branches of mathematics; and not more for the benefit of the student than for the progress of the science. Besides the main purpose of this work, the student will find many valuable, though incidental, aids in comprehending some of the more abstruse points in the subject, which are brought out in criticisms of some of the works in most common use. We append a large portion of the Author's Preface, as a far clearer and more critical analysis of the contents and spirit of the work, than the time now at our disposal will allow us to prepare.

"In 1810 a work was published in Cambridge under the following title: *A Treatise on Isoperimetrical Problems and the Calculus of Variations By* ROBERT WOODHOUSE, *A. M, F. R. S., Fellow of Caius College, Cambridge.* This work details the history of the Calculus of Variations from its origin until the close of the eighteenth century, and has obtained a high reputation for accuracy and clearness. During the present century some of the most eminent mathematicians have endeavored to enlarge the boundaries of the subject, and it seemed probable that a survey of what had been accomplished would not be destitute of interest and value. Accordingly the present work has been undertaken, and a short account will now be given of its plan.

"As the early history of the Calculus of Variations had been already so ably written, it was unnecessary to go over it again; but it seemed convenient to commence with a short account of two works of LAGRANGE and a work of LACROIX, because they exhibit the state of the subject at the close of the eighteenth century; the first chapter is therefore devoted to these works of LAGRANGE and LACROIX. The notice of the two works of LAGRANGE is very brief, for in fact both of them were accessible to WOODHOUSE, and he has given a good account of all that LAGRANGE accomplished. The notice of the work of LACROIX is fuller, because the second edition of that work had not appeared when WOODHOUSE wrote; it was also necessary to indicate two important mistakes which occur in LACROIX, on account of their influence on the history of the subject; see Arts. 27 and 39.

"The second chapter contains an account of the treatises of DIRKSEN and OHM.

"The third chapter contains an account of a remarkable memoir by GAUSS, which affords the earliest example of the discussion of a problem involving the variation of a double integral with variable limits of integration.

"The fourth chapter contains an account of a memoir by POISSON on the Calculus of Variations. The great object of this memoir is to exhibit the variation of a double integral when the limits of integration are variable. The memoir is important in itself, and also from the fact that it may be considered to have led the way for those which were written by OSTROGRADSKY, DELAUNAY, CAUCHY, and SARRUS.

"The fifth chapter contains an account of a memoir by OSTROGRADSKY; this memoir was suggested by POISSON'S, and its object is to exhibit the variation of a multiple integral when the limits of the integration are variable.

"The Academy of Sciences at Paris proposed for their mathematical prize subject for 1842, the Variation of Multiple Integrals. The prize was awarded to a memoir by SARRUS, and honorable mention was made of a memoir by DELAUNAY. The memoir of DELAUNAY is analyzed in the sixth chapter, and the memoir of SARRUS in the eighth chapter; the seventh chapter analyzes a memoir by CAUCHY, in which the results obtained by SARRUS are presented under a slightly different form.

"Here that part of the present work terminates which treats of the variation of multiple integrals.

"The next three chapters treat of another branch of the subject, namely, the criteria which distinguish a maximum from a minimum; these criteria were exhibited in a remarkable memoir published by JACOBI in 1837, which has given rise to a series of commentaries and developments. The method of JACOBI is founded upon one originally given by LEGENDRE; accordingly, the ninth chapter first explains what LEGENDRE accomplished, and also what was added to his results by another mathematician, BRUNACCI, and then finishes with a translation of JACOBI'S memoir. The tenth chapter con-

tains an account of the commentaries and developments to which JACOBI's memoir gave rise. The eleventh chapter contains some miscellaneous articles which also bear upon JACOBI's memoir.

"The twelfth chapter contains an account of various memoirs which illustrate special points in the Calculus of Variations. The thirteenth chapter contains an account of three comprehensive treatises which discuss the whole subject. The fourteenth chapter gives a brief notice of all the other treatises on the subject which have come to the writer's knowledge.

"The fifteenth chapter notices various memoirs which have some slight connection with the subject. The sixteenth chapter notices various memoirs which relate principally to geometry, or differential equations, or mechanics, but the titles of which are suggestive of some relation to the Calculus of Variations.

"The seventeenth chapter gives the history of the theory of the conditions of integrability.

"The writer has endeavored to be simple and clear, and he hopes that any student who has mastered the elements of the subject will be able without difficulty to understand the whole of the work.

"It may appear at first sight that great disproportion exists between the spaces devoted to the various treatises and memoirs which are analyzed. The writer has not considered solely or chiefly the relative importance of these treatises and memoirs, but also the ease or difficulty of obtaining access to them; and thus a work of inferior absolute value may sometimes have obtained as long a notice as another of higher character when the latter could be procured far more readily than the former.

"In citing an independent work the title has usually been given in the original language of the work, but in citing a memoir which forms part of a scientific journal it has generally been considered sufficient to give an English translation of the title. Sometimes a mathematician has been named in the history before an account of his contributions to the subject has been given; in such a case by the aid of the index of names at the end of the volume it will be easy to find the place which contains the account. Occasionally in the course of the translation of a passage from a foreign memoir the present writer has inserted a remark of his own; this remark will be known by being enclosed within square brackets.

"The writer may perhaps be excused for stating that he has found the labor attendant on the production of this work far longer and heavier than he had anticipated. It would have been easy to have examined merely the introductions to the various treatises and memoirs, and thus to have compiled an account of what their respective authors proposed to effect; but the object of the present writer was more extensive. He wished to ascertain distinctly what had been effected, and to form some estimate of the manner in which it had been effected. Accordingly, unless the contrary is distinctly stated, it may be assumed that any treatise or memoir relating to the Calculus of Variations which is described in the present work has undergone thorough examination and study. This remark does not, however, apply to all the productions which are noticed in the last two chapters of this work.

"It will be found that in the course of the history numerous remarks, criticisms, and corrections are suggested relative to the various treatises and memoirs which are analyzed. The writer trusts that it will not be supposed that he undervalues the labors of the eminent mathematicians in whose works he ventures occasionally to indicate inaccuracies or imperfections, but that his aim has been to remove difficulties which might perplex a student. In the course of his studies the writer frequently found that remarks which he intended to offer on various points had been already made by some author not usually consulted; for example, the considerations introduced in Art. 366 occurred to him at the commencement of his studies, and it was not until long afterwards that he found he had been anticipated by LEGENDRE; see Art. 202.

"The writer will not conceal his own opinion of the value of a history of any department of science when that history is presented with accuracy and completeness. It is of importance that those who wish to improve or extend any subject should be able to ascertain what results have already been obtained, and thus reserve their strength for difficulties which have not yet been overcome; and those who merely desire to ascertain the present state of a subject without any purpose of original investigation will often find that the study of the past history of that subject assists them materially in obtaining a sound and extensive knowledge of the position to which it has attained. How far the present work deserves attention must be left to competent judges to decide; should they consider that the objects proposed have been in some degree secured, the writer will be encouraged hereafter to undertake a similar survey of some other department of science."

It gives us great pleasure to thank the author for this valuable and greatly needed work, and to express the sincere wish that he may be encouraged to pursue a line of research for which the volume before us shows him so admirably qualified.

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ON THE RESULTS OF ACCELERATED VELOCITY.

By JOHN PATERSON, A. M., Albany, N. Y.

I. IF a material unit, 1_μ, move uniformly (*in vacuo*) in time, and 1_t be the distance it travels or describes in the assumed unit of time 1_t, it will describe the distance $x\,1_t$ in the time $x\,1_t$: the velocity is constant; its unit may be written 1_λ, which is its measure as a cause, and is equal to 1_t, the measure of its effect in the unit of time 1_t; and the distance the constant velocity generates increases uniformly as the time, being equal to the product of the velocity by the time $1_\lambda \,.\, x\,1_t = x\,1_t$.

II. If the velocity increase uniformly in time (beginning at zero), and $m\,1_\lambda$ be the amount accruing in the first unit of time 1_t (m to be determined), this amount will become $m\,x\,1_\lambda$ in the time $x\,1_t$. At the expiration of the time $x\,1_t$ let the increase cease, and the velocity remain constant with its ultimate value acquired in this time. With this ultimate constant value, the velocity will carry the material unit or body 1_μ some number, m, of times the distance in the succeeding time $x\,1_t$ that it carried the same body in the past equal time $x\,1_t$, while increasing from zero to its ultimate value, $m\,x\,1_\lambda$; that is, if l be the distance described or generated in the past time $x\,1_t$, $m\,l$ will be the distance generated in the future equal time $x\,1_t$, and, conversely, the former distance will be the mth part of the latter; and

it is obvious that l may be taken for unit of distance 1_l. Thus $m x 1_\lambda$ expresses the amount of velocity (now to remain constant) acquired by 1_μ in the first interval $x 1_t$ of time. Since $1_\lambda = 1_l$, this velocity will carry 1_μ the distance $m x 1_l$ in the succeeding unit of time 1_t, and the distance $m x^2 1_l$ in the time $x 1_t$: consequently the mth part of this distance, $= x^2 1_l$, is the distance described by 1_μ while the velocity was increasing uniformly from zero to the ultimate value $m x 1_\lambda$ in the past time $x 1_t$.

Divide the time into any two (equal or unequal) intervals, $\frac{1}{n} x 1_t$ and $\frac{n-1}{n} x 1_t$. As the final velocity has increased uniformly in time, it is $\frac{m}{n} x 1_\lambda$ and $\frac{(n-1)m}{n} x 1_\lambda$ at the expiration of these respective intervals of time. The velocity $\frac{m}{n} x 1_\lambda$ would carry 1_μ the distance $\frac{m}{n} x^2 1_l$ in the time $x 1_t$, which reduces to $\frac{m(n-1)}{n^2} x^2 1_l$ for the interval $\frac{n-1}{n} x 1_t$ (being the product of the velocity by the time), and to $\frac{1}{n^2} x^2 1_l$ for the past interval $\frac{1}{n} x 1_t$ (being the mth part of the product of the ultimate velocity by the time). The velocity $\frac{m(n-1)}{n} x 1_\lambda$ would carry 1_μ the distance $\frac{m(n-1)}{n} x^2 1_l$ in the time $x 1_t$, which reduces to $\frac{m(n-1)^2}{n^2} x^2 1_l$ for the subsequent interval $\frac{n-1}{n} x 1_t$, and to $\left(\frac{n-1}{n}\right)^2 x^2 1_l$ for the interval $\frac{n-1}{n} x 1_t$ during which the ultimate value was acquired. Since $x^2 1_l$ is the whole distance described by 1_μ in the entire interval $x 1_t$ of past time, there has arisen the equation

$$\left(\frac{1}{n^2} x^2 + \frac{m(n-1)}{n^2} x^2 + \left(\frac{n-1}{n}\right)^2 x^2\right) 1_l = x^2 1_l;$$

whence $m = 2$.

Hence, when a velocity increases uniformly, its ultimate value for the time $x 1_t$ is $2 x 1_\lambda$, and $x^2 1_l$ is the distance it has generated while acquiring this ultimate value; that is, when the velocity increases as the time, the distance increases as the square of the time; or, the

increase of the velocity is uniform, and the movement is uniformly accelerated.

III. Suppose, next, that the increase of the velocity is uniform, or that the velocity is uniformly accelerated, and therefore increases as the square of the time. If l be the distance generated in the past time $x\,1_t$, the ultimate velocity, become constant, will generate the distance $m'l$ in the equal future time $x\,1_t$ (m' to be determined). But $m'x^2\,1_\lambda$, being the ultimate velocity, will generate the distance $m'x^2\,1_t$ in the unit of time 1_t, and $m'x^3\,1_t$ in the time $x\,1_t$: consequently $x^3\,1_t$ was generated in the past time $x\,1_t$.

Dividing the time $x\,1_t$ into two intervals, as before, the entire right-hand member of the equation in the preceding paragraph is constituted of three partial terms in the left-hand member,

$$\frac{1}{n^2}x^2 + \frac{2(n-1)}{n^2}x^2 + \left(\frac{n-1}{n}\right)^2 x^2;$$

and now, instead of increments of distance, these several terms represent increments of velocity, each increasing from zero to an ultimate value, to be expressed by introducing the factor m'.

Now $\frac{m'}{n^2}x^2\,1_\lambda$ would generate $\frac{m'}{n^2}x^2\,1_t$ in the unit of time 1_t, and therefore $\frac{m'}{n^2}x^2\,1_t$ in the time $x\,1_t$, which reduces to $\frac{m'(n-1)}{n^3}x^3\,1_t$ for the time $\frac{n-1}{n}x\,1_t$ and to $\frac{1}{n^3}x^3\,1_t$ for the time $\frac{1}{n}x\,1_t$; and, similarly, $\frac{m'(n-1)^2}{n^2}x^2\,1_\lambda$ generates $\frac{(n-1)^3}{n^3}x^3\,1_t$ in the time $\frac{n-1}{n}x\,1_t$. Each of these increments of velocity is *uniformly accelerated* from zero to the ultimate value expressed by m'; but the increment $\frac{2m'(n-1)}{n^2}x^2\,1_\lambda$ only increases uniformly from zero to its ultimate value during the time $\frac{n-1}{n}x\,1_t$; and hence, while, with its ultimate value constant, it would generate the distance $\frac{2m'(n-1)}{n^2}x^3\,1_t$ in the time $x\,1_t$, this distance first reduces to $\frac{2m'(n-1)^2}{n^3}x^3\,1_t$ for the subsequent time $x\,1_t$, and then to $\frac{2}{2}\cdot\frac{m'(n-1)^2}{n^3}x^3\,1_t$ for the time $\frac{n-1}{n}x\,1_t$ during which the

increment acquired its ultimate value by uniform increase. Hence appears the equation

$$\left(\frac{1}{n^3}x^3 + \frac{m'(n-1)}{n^3}x^3 + \frac{m'(n-1)^2}{n^3}x^3 + \frac{(n-1)^3}{n^3}x^3\right)1_t = x^3 1_t,$$

which gives $m' = 3$. Consequently, when the velocity is uniformly accelerated, or increases in duplicate ratio of the time, its ultimate value is $3x^2 1_\lambda$; which, being constant, would carry 1_μ the distance $3x^3 1_t$ in the coming time $x1_t$, one third of which is $1x^3 1_t$, the distance 1_μ was carried in the past equal time $x1_t$: the distance increases as the cube of the time (in triplicate ratio).

IV. If the velocity be duplicately accelerated, or increase in triplicate ratio of the time, the preceding example prescribes the partial velocities

$$\frac{m''}{n^3}x^3 1_\lambda + \frac{3m''(n-1)}{n^3}x^3 1_\lambda + \frac{3m''(n-1)^2}{n^3}x^3 1_\lambda + \frac{m''(n-1)^3}{n^3}x^3 1_\lambda;$$

the first and last terms of which increase triplicately in the respective times $\frac{1}{n}x1_t$ and $\frac{n-1}{n}x1_t$, and, reasoning similarly as in the foregoing cases, their results are $\frac{1}{n^4}x^4 1_t$ in $\frac{1}{n}x1_t$, and $\frac{m''(n-1)}{n^4}x^4 1_t$ and $\frac{(n-1)^4}{n^4}x^4 1_t$ in $\frac{n-1}{n}x1_t$: the second term increases uniformly in the time $\frac{n-1}{n}x1_t$, and therefore its result is $\frac{3}{2}\frac{m''(n-1)^2}{n^4}x^4 1_t$ in that time; and the third term increases duplicately, and its result is $\frac{3}{3}\frac{m''(n-1)^3}{n^4}x^4 1_t$ in the time $\frac{n-1}{n}x1_t$. Arrangement and addition show

$$\frac{1}{n}x1_t + \frac{n-1}{n}x1_t = 1x1_t.$$

$$\left.\begin{aligned}\frac{1}{n^4}x^4 1_t &+ \frac{m''(n-1)}{n^4}x^4 1_t \\ &+ \frac{3}{2}\frac{m''(n-1)^2}{n^4}x^4 1_t \\ &+ \frac{3}{3}\frac{m''(n-1)^3}{n^4}x^4 1_t \\ &+ \frac{(n-1)^4}{n^4}x^4 1_t\end{aligned}\right\} = 1x^4 1_t,$$

which gives $m'' = 4$. The result increases as the biquadrate of the time.

The same method of investigation applies to cases where the velocity is triplicately, quadruplicately, etc. accelerated. The *a priori* or deductive method, however, gives at once the complete demonstration of the genesis of the entire series or hierarchy of corresponding simultaneous and successive powers (or faculties) denoted as follows: —

$$1\varphi^n,\ 1x\varphi^{n-1},\ 1x^2\varphi^{n-2},\ 1x^3\varphi^{n-3},\ \ldots\ 1x^{n-3}\varphi''',\ 1x^{n-2}\varphi'',\ 1x^{n-1}\varphi',\ 1x^n\varphi^0,$$

and

$$1\varphi^n,\ 2x\varphi^{n-1},\ 3x^2\varphi^{n-2},\ 4x^3\varphi^{n-3},\ \ldots\ (n-2)x^{n-3}\varphi''',\ (n-1)x^{n-2}\varphi'',\ nx^{n-1}\varphi',\ 1x^n\varphi^0,$$

which govern the entire theory of development, and exhibit the actual composition of the differential and integral coefficients and the origin of factorials. But inasmuch as this demonstration has heretofore been published, and I believe has failed to excite any attention, I have prepared the present article more in accordance with the methods of induction, and will be thankful for an exposure of its fallacy.

FOURTH PRIZE ESSAY.—SPHERICAL CONICS.

By C. M. Woodward, Newburyport, Mass.

1. It is the object of this paper to give some equations, and to investigate the nature, of the intersections made by a cone on the surface of a sphere whose centre is at the vertex. The cone may be any cone of the second degree; i. e. it may have any section of the second degree for its base.

Let C be the centre of the sphere, and O the foot of the perpendicular dropped from C to the base of the cone; and let $CO =$ the

radius of the sphere = unity, so that the base may be tangent to the sphere at O. Now, let O be the origin of a system of rectangular co-ordinates, and let

$$f(x, y) = 0 \tag{1}$$

be the equation of the base of the cone. The equations for transforming (1) to a system of polar co-ordinates, the origin being the same, and the axis of X being the polar axis, are $x = r\cos\varphi, y = r\sin\varphi$. If we let ϱ be the arc of the sphere corresponding to the radius vector r, so that $r = \tan\varrho$, these values of x and y become

$$\left.\begin{aligned} x &= \tan\varrho\ \cos\varphi, \\ y &= \tan\varrho\ \sin\varphi, \end{aligned}\right\} \tag{2}$$

which, substituted in (1), give

$$f(\tan\varrho\ \cos\varphi,\ \tan\varrho\ \sin\varphi) = 0. \tag{3}$$

Now, if the curve represented by the base of the cone be projected upon the sphere on right lines drawn towards its centre, it will evidently become the curve of the intersection of the cone and the sphere; i. e. it will be a spherical conic. But ϱ, the spherical radius vector, and φ, the polar angle, are the independent variables of the conic; and certain functions of these, derived from (2), must be substituted for x and y in the rectangular equations of the base, in order to give the equations of the spherical conic in terms of ϱ and φ.

2. It is evident, from the nature of the sphere and cone, that whatever may be the section of the cone made by its base, or, in other words, wherever we take the point O, at which point the base is tangent to the sphere, the projection of the plane curve always gives the *same spherical curve*. And it will be shown that, as the plane section of the cone is an ellipse, an hyperbola, or a parabola, according to the position of the point O, so the spherical curve, when viewed from the same positions, will appear to be in turn a spherical

ellipse, a spherical hyperbola, and a spherical parabola, though the curve itself remains in all cases the same.

3. *Equation of the Spherical Ellipse.* — Let us now take the general equation of a cone,

$$\frac{x^2}{A^2} + \frac{y^2}{B^2} - \frac{z^2}{C^2} = 0, \tag{4}$$

and find from it the equation of a section parallel to the plane xy, and at a distance from it equal to unity. Substituting $z = 1$ in (4), it becomes $\frac{C^2x^2}{A^2} + \frac{C^2y^2}{B^2} = 1$, and if we make $\frac{A}{C} = a$, $\frac{B}{C} = b$, whence

$$\frac{A}{B} = \frac{a}{b}, \tag{5}$$

it reduces to the form $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \tag{6}$$

which is the equation of an ellipse, the origin being at the centre and the axes $2a$ and $2b$. Since the axis of z is the axis of the cone, the point O is the centre of the ellipse.

Equation (6) becomes, by substituting (2),

$$\frac{\tan^2\varrho\cos^2\varphi}{a^2} + \frac{\tan^2\varrho\sin^2\varphi}{b^2} = 1,$$

or,
$$\frac{\cos^2\varphi}{a^2} + \frac{\sin^2\varphi}{b^2} = \frac{1}{\tan^2\varrho}. \tag{7}$$

Now, if $\varphi = 0°$, ϱ coincides with the transverse axis of the spherical curve, which we will call 2α, and (7) gives

$$\frac{1}{a^2} = \frac{1}{\tan^2\varrho}, \qquad \text{or,} \qquad a = \tan\varrho = \tan\alpha. \tag{8}$$

Also, if we make $2\beta =$ the conjugate axis when $\varphi = 90°$, (7) gives

$$\frac{1}{b^2} = \frac{1}{\tan^2\varrho}, \qquad \text{or,} \qquad b = \tan\varrho = \tan\beta. \tag{9}$$

Substituting these values of a and b in (7), we get

$$\frac{\cos^2\varphi}{\tan^2\alpha} + \frac{\sin^2\varphi}{\tan^2\beta} = \frac{1}{\tan^2\varrho}, \tag{10}$$

which is the equation of a spherical ellipse referred to its principal diameters.

4. *Discussion of* (10). — Since the first member of (10) is always positive, φ may have any value, and the curve extends completely around the origin. But as the cone has two nappes, its intersections with the sphere will give two equal spherical curves, which are diametrically opposite to each other; so that the equation, to be complete, must be the equation of both curves. Now the tangent of any angle is the same as the tangent of the angle plus 180°, or π; hence every value of $\tan\varrho$ gives two values of the radius vector differing from each other by a semi-circumference. Thus, when $\varphi = 0°$, (10) gives $\tan\varrho = \pm\tan\alpha$, $\varrho = \pm\alpha$, or $\pm(\pi+\alpha)$, determining two points on each curve. Therefore (10) is sufficient for both curves, even if we neglect the negative values of $\tan\varrho$.

5. *Equation of the Spherical Hyperbola.* — If we substitute in (4) the value $x = 1$, we shall get the equation of the section made by a plane parallel to the plane yz, and tangent to the sphere. Since this plane is perpendicular to the plane considered in § 3, O', the point of tangency to the sphere and the centre of the hyperbola, is 90° from the point O, which was the centre of the ellipse. In this case the resulting equation is $\frac{A^2z^2}{C^2} - \frac{A^2y^2}{B^2} = 1$, which by (5) becomes

$$a^2z^2 - \frac{a^2}{b^2}y^2 = 1, \qquad (11)$$

which is the equation of a plane hyperbola referred to its principal diameters. Substituting in (11) the values of y and z derived from (2), (for z can obviously take the place of x,) and representing in this case the radius vector and the polar angle by ϱ_1 and φ_1, we get

$$a^2\tan^2\varrho_1\cos^2\varphi_1 - \frac{a^2}{b^2}\tan^2\varrho_1\sin^2\varphi_1 = 1,$$

$$a^2\cos^2\varphi_1 - \frac{a^2}{b^2}\sin^2\varphi_1 = \frac{1}{\tan^2\varrho_1}. \qquad (12)$$

Now if we denote the semi-axes of this curve by α_1 and β_1, when $\varphi_1 = 0°$, (12) gives $a^2 = \frac{1}{\tan^2 \varrho_1}$, or $a = \cot \varrho_1 = \cot \alpha_1 = \tan \alpha$; thus showing that α and α_1 are complements of each other. Since the semi-conjugate axis of the plane hyperbola is $\frac{b}{a}$, we shall have $\tan \beta_1 = \frac{b}{a} = \frac{\tan \beta}{\tan \alpha}$. Hence we get from (12) the equation

$$\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} - \frac{\sin^2 \varphi_1}{\tan^2 \beta_1} = \frac{1}{\tan^2 \varrho_1}, \qquad (13)$$

which is the equation of a spherical hyperbola, the origin being at the centre, and the axes being $2\alpha_1$ and $2\beta_1$.

6. *Discussion of* (13). — It is evident from the nature of the equation that ϱ_1 will be real for every value of φ_1, which makes $\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} > \frac{\sin^2 \varphi_1}{\tan^2 \beta_1}$, and imaginary for all other values.

The limits to the value of φ_1 are determined by the equation $\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} = \frac{\sin^2 \varphi_1}{\tan^2 \beta_1}$, which gives

$$\tan^2 \varphi_1 = \frac{\tan^2 \beta_1}{\tan^2 \alpha_1} = \tan^2 \beta,$$

or, $\tan \varphi_1 = \pm \tan \beta$, $\varphi_1 = \pm \beta$, or, $\pm (\pi + \beta)$.

Hence it is plain that for any value of φ_1 between β and $\pi - \beta$, and between $-\beta$ and $-(\pi - \beta)$, we shall have $\tan^2 \varphi_1 > \tan^2 \beta$, or $\frac{\cos^2 \varphi_1}{\tan^2 \alpha_1} < \frac{\sin^2 \varphi_1}{\tan^2 \beta_1}$; so that ϱ_1 will be real only for those values of φ_1 between β and $-\beta$, and between $\pi - \beta$ and $\pi + \beta$. When $\varphi_1 = 0°$, (13) gives

$$\tan \varrho_1 = \pm \tan \alpha_1 \quad \text{and} \quad \varrho_1 = \pm \alpha_1, \quad \text{or,} \quad = \pm (\pi + \alpha_1),$$

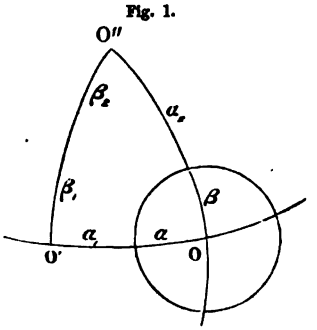
so that every value of φ_1 for which ϱ_1 is real gives four points, two on each branch of the curve. Hence, as in the ellipse, the negative sign may be neglected. It is evident that this curve lacks one of the essential characteristics of the plane hyperbola, viz. its infinite branches; and that the curve on each branch, on reaching a point 90° from the origin, returns symmetrically into itself.

7. *Equation of Spherical Hyperbola referred to another Set of Axes.* — If we take a section of the cone parallel to the plane xz, letting $y = 1$, we shall get in a similar manner the equation

$$b^2 z^2 - \frac{b^2}{a^2} x^2 = 1, \tag{14}$$

which is the equation of an hyperbola lying in a plane perpendicular to that expressed by (13). Since O', the point of tangency in this case, is the pole of the arc OO', the transverse axis α_2 of the curve is the complement of the conjugate axis β of the ellipse, and the conjugate axis β_2 is the complement of β_1 of the other hyperbola, as is evident from Fig. 1.

Fig. 1.



Hence we have $\tan \alpha_2 = \cot \beta = \frac{1}{b}$, and $\tan \beta_2 = \cot \beta_1 = \frac{\tan \alpha}{\tan \beta} = \frac{a}{b}$. The substitution of these values in (14), together with the values of z and x derived from (2) by putting z for x and x for y, and using ρ_2 and φ_2, gives

$$\frac{\cos^2 \varphi_2}{\tan^2 \alpha_2} - \frac{\sin^2 \varphi_2}{\tan^2 \beta_2} = \frac{1}{\tan^2 \rho_2}, \tag{15}$$

which is the equation of a spherical hyperbola (Fig. 1), its centre being the origin, and its axes being $2\alpha_2$ and $2\beta_2$. The discussion of (15) is similar to that of § 6.

8. *Equation of Spherical Parabola.* — Let us now transform equation (4) to a new system of rectangular co-ordinates, whose axes are X_1, Y_1, and Z_1, having the same origin, the axis of Y_1 coinciding with Y, and the axes of X_1 and Z_1 making the angle α with X and Z respectively. It is evident, since α is half the angle at the vertex of the cone in the plane xz, that the new axis Z_1 coincides with the side of the cone, and the section made by a plane parallel to $y_1 z_1$ will be a

parabola. On these conditions the equations for transformation are $x = x_1 \cos\alpha - z_1 \sin\alpha$, $y = y_1$, $z = x_1 \sin\alpha + z_1 \cos\alpha$. Substituting these values in (4), it becomes

$$\frac{x_1^2\cos^2\alpha - 2x_1z_1\cos\alpha\sin\alpha + z_1^2\sin^2\alpha}{A^2} + \frac{y_1^2}{B^2} - \frac{x_1^2\sin^2\alpha + 2x_1z_1\sin\alpha\cos\alpha + z_1^2\cos^2\alpha}{C^2} = 0,$$

$$y_1^2 = \frac{B^2}{C^2}(x_1^2\sin^2\alpha + 2x_1z_1\sin\alpha\cos\alpha + z_1^2\cos^2\alpha) - \frac{B^2}{A^2}(x_1^2\cos^2\alpha - 2x_1z_1\sin\alpha\cos\alpha + z_1^2\sin^2\alpha).$$

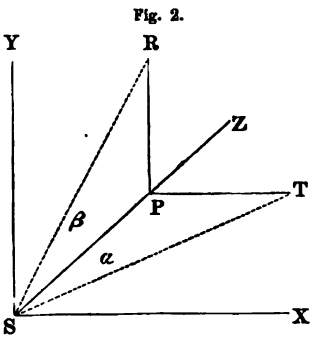
If we make $x_1 = 1$, the above equation becomes the equation of a section made by a plane parallel to y_1z_1, and tangent to the sphere. This substitution gives

$$y_1^2 = \frac{B^2}{C^2}(\sin^2\alpha + z_1\sin 2\alpha + z_1^2\cos^2\alpha) - \frac{B^2}{A^2}(\cos^2\alpha - z_1\sin 2\alpha + z_1^2\sin^2\alpha),$$

$$= \frac{B^2}{C^2}\sin^2\alpha - \frac{B^2}{A^2}\cos^2\alpha + \sin 2\alpha\left(\frac{B^2}{C^2} + \frac{B^2}{A^2}\right)z_1 + \left(\frac{B^2}{C^2}\cos^2\alpha - \frac{B^2}{A^2}\sin^2\alpha\right)z_1^2. \quad (16)$$

Let P (Fig. 2) be any point in the axis of Z, and draw PT parallel to SX, and PR parallel to SY, so that T and R may be at the surface of the cone. Then the lines PT, PR, and SP represent the ratios which the quantities A, B, and C bear to each other in the equation of the cone. Also, the angle $TSP = \alpha$ and $RSP = \beta$.

Fig. 2.



Hence $\cos\alpha = \frac{SP}{ST} = \frac{SP}{\sqrt{((SP)^2 + (PT)^2)}}$,

$$\cos^2\alpha = \frac{SP^2}{SP^2 + PT^2} = \frac{C^2}{C^2 + A^2},$$

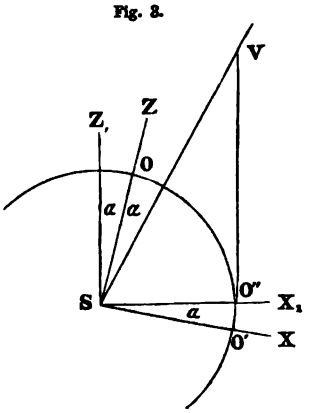
also, $$\sin^2\alpha = \frac{A^2}{C^2 + A^2}.$$

Substituting these values in the coefficient of z_1^2 in (16), it becomes $\frac{B^2}{C^2} \times \frac{C^2}{A^2 + C^2} - \frac{B^2}{A^2} \times \frac{A^2}{A^2 + C^2} = 0$; so that the term containing z_1^2 disappears, and (16) becomes

$$y_1^2 = \frac{B^2}{C^2}\sin^2\alpha - \frac{B^2}{A^2}\cos^2\alpha + \sin 2\alpha\left(\frac{B^2}{C^2} + \frac{B^2}{A^2}\right)z_1, \quad (17)$$

which is the rectangular equation of a parabola lying in the plane $y_1 z_1$, the axis of z_1 being the axis of the curve, and the distance of the vertex from the origin being

Fig. 3.



$$\frac{\frac{B^2}{A^2}\cos^2\alpha - \frac{B^2}{C^2}\sin^2\alpha}{\sin 2\alpha\left(\frac{B^2}{C^2}+\frac{B^2}{A^2}\right)}. \qquad (18)$$

Let O'' (Fig. 3) be the point at which the cutting plane is tangent to the sphere, and let V be the vertex of the parabola. Then obviously

$$O''V = \tan(\tfrac{1}{2}\pi - 2\alpha) = \cot 2\alpha = (18).$$

This last equation is shown to be true by reducing (18) and $\cot 2\alpha$ to the same form. Thus, by substituting values for $\cos^2\alpha$ and $\sin^2\alpha$, (18) becomes

$$\frac{\frac{B^2}{A^2}\left(\frac{C^2}{A^2+C^2}\right) - \frac{B^2}{C^2}\left(\frac{A^2}{A^2+C^2}\right)}{\left(\frac{B^2}{C^2}+\frac{B^2}{A^2}\right)\frac{2AC}{A^2+C^2}} = \frac{\frac{C^2}{A^2}-\frac{A^2}{C^2}}{\frac{2A}{C}+\frac{2C}{A}} = \frac{\frac{C^4-A^4}{AC}}{2(A^2+C^2)} = \frac{C^2-A^2}{2AC}. \qquad (19)$$

Also, from (5) and (8) we have

$$\cot 2\alpha = \frac{1-\tan^2\alpha}{2\tan\alpha} = \frac{1-\frac{A^2}{C^2}}{\frac{2A}{C}} = \frac{\frac{C^2-A^2}{C^2}}{\frac{2A}{C}} = \frac{C^2-A^2}{2AC} = (19). \quad \text{Q. E. D.}$$

Substituting in (17) values of y_1 and z_1 taken from (2), and making use of (5), (8), and (9), we have

$$\tan^2\rho\sin^2\varphi = \tan^2\beta\sin^2\alpha - \frac{\tan^2\beta}{\tan^2\alpha}\cos^2\alpha + \sin 2\alpha\left(\tan^2\beta + \frac{\tan^2\beta}{\tan^2\alpha}\right)\tan\rho\cos\varphi,$$

$$\frac{\tan^2\rho\sin^2\varphi}{\tan^2\beta} = \frac{\sin^4\alpha - \cos^4\alpha}{\sin^2\alpha} + 2\cos\alpha\sin\alpha\operatorname{cosec}^2\alpha\tan\rho\cos\varphi,$$

$$\frac{\tan^2\rho\sin^2\varphi}{\tan^2\beta} = 1 - \cot^2\alpha + 2\cot\alpha\tan\rho\cos\varphi, \qquad (20)$$

which is the equation of a spherical parabola, α being on the axis of the curve, and the distance of the vertex from the origin being $\frac{1}{2}\pi - 2\alpha$.

9. *Discussion of* (20).— From (20) we get, by solving the quadratic equation,

$$\tan\varrho = \cot\alpha \tan^2\beta \frac{\cot\varphi}{\sin\varphi} \pm \sqrt{\left(\frac{\tan^2\beta(1-\cot^2\alpha) + \cot^2\alpha \tan^4\beta \cot^2\varphi}{\sin^2\varphi}\right)}; \quad (21)$$

so that here, as in the other curves, we get four values of ϱ for every value of φ for which $\tan\varrho$ is real. When $\varphi = 0$, (20) gives

$$\tan\varrho = \frac{\cot^2\alpha - 1}{2\cot\alpha} = \frac{\tan^2(\frac{1}{2}\pi - \alpha) - 1}{2\tan(\frac{1}{2}\pi - \alpha)} = -\operatorname{cotan}(\pi - 2\alpha)$$

$$= \tan(\tfrac{1}{2}\pi - 2\alpha), \quad \varrho = \tfrac{1}{2}\pi - 2\alpha, \quad \text{or,} \quad = \tfrac{3}{2}\pi - 2\alpha,$$

which gives the points at the vertices of the two curves; for evidently this equation is the equation of the two intersections on the sphere, thus requiring two plane parabolas to make one spherical parabola.

The limits for the value of φ are determined by the equation $1 - \cot^2\alpha + \cot^2\alpha \tan^2\beta \cot^2\varphi = 0$, taken from the radical in (21). This equation gives

$$\cot^2\varphi = \frac{\cot^2\alpha - 1}{\cot^2\alpha \tan^2\beta} = \frac{1 - \tan^2\alpha}{\tan^2\beta} = \cot^2\beta - \cot^2\beta_1$$

$$= \frac{C^2}{B^2} - \frac{A^2}{B^2} = \frac{C^2 - A^2}{B^2}, \qquad \text{or,} \qquad \tan\varphi = \pm\frac{B}{\sqrt{(C^2 - A^2)}}.$$

Hence, $\varphi = \tan^{[-1]}\left[\pm\frac{B}{\sqrt{(C^2 - A^2)}}\right]$, or, $\pm\left(\pi + \tan^{[-1]}\left[\frac{B}{\sqrt{(C^2 - A^2)}}\right]\right)$.

So that ϱ is imaginary for all values of φ between

$$\tan^{[-1]}\left(\frac{B}{\sqrt{(C^2 - A^2)}}\right) \quad \text{and} \quad \pi - \tan^{[-1]}\left(\frac{B}{\sqrt{(C^2 - A^2)}}\right),$$

and between

$$\tan^{[-1]}\left(-\frac{B}{\sqrt{(C^2 - A^2)}}\right) \quad \text{and} \quad \pi + \tan^{[-1]}\left(\frac{B}{\sqrt{(C^2 - A^2)}}\right),$$

and real for all others.

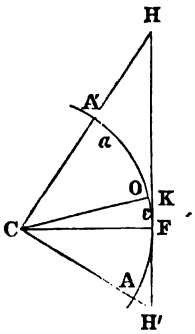
Since here, as in all the other curves, the absolute value of ρ is the same whether φ is positive or negative, the two branches of the curve must be symmetrical.

10. From these investigations, it is evident that any conic section, made by a plane tangent to the sphere whose centre is at the vertex, may be projected on the sphere in the shape of a spherical conic. Moreover, although the curves themselves remain the same, the peculiar characteristics which they assume as a spherical conic — i. e. whether an ellipse, parabola, or hyperbola — are determined solely by the position of the point from which we start as the origin, and at which the cutting plane is tangent.

It must be observed, that, though the ellipse and parabola, when projected on the sphere, give in each case a complete spherical oval, this is not true of each branch of the hyperbola. Each branch, when projected on its own side alone, gives only a portion of the curve, but these portions are complements of each other. But since the tan ρ for any point is the same as for the point diametrically opposite, each curve is twice projected, and thus we get both the curves actually made by the intersections on the sphere.

11. Both the spherical curves we have considered are of double curvature, except when in (4) $A = B = C$, in which case the cone is a right cone, the ellipses are circles, both the hyperbolas are equilateral, and the origin of the parabola, O'', is midway between O' and O.

Fig. 4.



12. *Equation of the Spherical Ellipse referred to the Focus.* — Let us now suppose that the cutting plane of the ellipse is so turned that the point of contact with the sphere is one of the foci. Then F (Fig. 4) is the focus, and $AO = A'O = a$. Denote the arc FO by ε, and let HH' be the greater axis of the ellipse. Then

$$FH = \tan(\alpha + \varepsilon), \qquad FH' = \tan(\alpha - \varepsilon),$$

and
$$HH' = 2a = \tan(\alpha + \varepsilon) + \tan(\alpha - \varepsilon)$$

$$= \frac{\sin(\alpha+\varepsilon)\cos(\alpha-\varepsilon) + \sin(\alpha-\varepsilon)\cos(\alpha+\varepsilon)}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)} = \frac{2\sin\alpha\cos\alpha}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)};$$

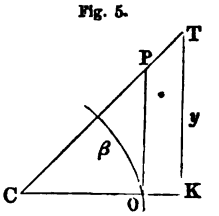
$$\therefore a = \frac{\sin\alpha\cos\alpha}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}. \tag{22}$$

If we represent by m and n the segments of the transverse axis made by any ordinate y, in a plane ellipse, we have, from the well-known properties of the curve, $\frac{y^2}{mn} = \frac{b^2}{a^2}$. Let $m = KH'$ (Fig. 4) and $n = KH$; then

$$\left.\begin{aligned} m &= \tan\varepsilon + \tan(\alpha - \varepsilon) = \frac{\sin\alpha}{\cos\varepsilon\cos(\alpha-\varepsilon)}, \\ \text{and} \qquad n &= \tan(\alpha+\varepsilon) - \tan\varepsilon = \frac{\sin\alpha}{\cos\varepsilon\cos(\alpha+\varepsilon)}. \end{aligned}\right\} \tag{23}$$

Now, since y is perpendicular to CO, it must be in the same plane with the semi-conjugate axis β, and we have from the similar triangles in Fig. 5 the proportion $CO : CK = OP : KT$. But $CO = 1$, $CK = \sec\varepsilon$, $OP = \tan\beta$, and $KT = y$, and we have $y = \tan\beta\sec\varepsilon$. Therefore, by (23),

Fig. 5.



$$\frac{y^2}{mn} = \frac{\tan^2\beta\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}{\sin^2\alpha} = \frac{b^2}{a^2}.$$

But by (22)

$$\frac{b^2}{a^2} = \frac{b^2\cos^2(\alpha+\varepsilon)\cos^2(\alpha-\varepsilon)}{\sin^2\alpha\cos^2\alpha} = \frac{\tan^2\beta\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}{\sin^2\alpha};$$

hence,
$$b^2 = \frac{\tan^2\beta\cos^2\alpha}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}. \tag{24}$$

The polar equation of the plane ellipse, the origin being at one focus, and the axis directed from the other, is $r = \frac{b^2}{a + c\cos\varphi}$; hence

$$\frac{1}{r} = \cot\rho = \frac{a}{b^2} + \frac{c}{b^2}\cos\varphi. \tag{25}$$

From (Fig. 4), $c = \frac{1}{2}(HF - H'F) = \frac{1}{2}(\tan(\alpha+\varepsilon) - \tan(\alpha-\varepsilon))$

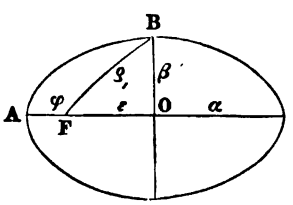
$$= \frac{\sin\varepsilon\cos\varepsilon}{\cos(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}. \qquad (26)$$

The substitution of (22), (24), and (26) in (25) gives

$$\cot\rho = \frac{\sin\alpha\cos\alpha + \sin\varepsilon\cos\varepsilon\cos\varphi}{\tan^2\beta\cos^2\alpha}, \qquad (27)$$

which is the equation of a spherical ellipse, the origin being at one focus, and the polar axis directed from the other.

Fig. 6.



13. *To prove that* ρ_1, *the radius vector drawn to the extremity of the conjugate axis, equals* α. — Let Fig. 6 represent the spherical ellipse, and we have, by NAPIER'S Rule,

$$\cos\varphi = -\tan\varepsilon\cot\rho_1,$$

which, substituted in (27), gives

$$\cot\rho_1 = \frac{\sin\alpha\cos\alpha - \sin^2\varepsilon\cot\rho_1}{\tan^2\beta\cos^2\alpha} = \frac{\cos\alpha\sin\alpha}{\tan^2\beta\cos^2\alpha + \sin^2\varepsilon}. \qquad (28)$$

Now when $\varphi = 0$, $\rho = \alpha - \varepsilon$, and (27) becomes

$$\cot(\alpha-\varepsilon) = \frac{\sin\alpha\cos\alpha + \cos\varepsilon\sin\varepsilon}{\tan^2\beta\cos^2\alpha} = \frac{\sin 2\alpha + \sin 2\varepsilon}{2\tan^2\beta\cos^2\alpha}$$

$$= \frac{\sin(\alpha+\varepsilon)\cos(\alpha-\varepsilon)}{\tan^2\beta\cos^2\alpha} = \frac{\cos(\alpha-\varepsilon)}{\sin(\alpha-\varepsilon)}; \qquad (29)$$

hence, $\sin(\alpha+\varepsilon)\sin(\alpha-\varepsilon) = \tan^2\beta\cos^2\alpha.$

By trigonometry, $\sin(\alpha+\varepsilon)\sin(\alpha-\varepsilon) = \frac{1}{2}(\cos(2\varepsilon) - \cos 2\alpha)$;

and since $\cos 2\alpha = 2\cos^2\alpha - 1$, and $\cos 2\varepsilon = 2\cos^2\varepsilon - 1$,

we have $\cos^2\varepsilon - \cos^2\alpha = \tan^2\beta\cos^2\alpha,$

or, $\cos^2\varepsilon = \cos^2\alpha + \tan^2\beta\cos^2\alpha;$ (30)

hence, $\sin^2\varepsilon = 1 - \cos^2\varepsilon = 1 - \cos^2\alpha - \tan^2\beta\cos^2\alpha,$

which, substituted in (28), gives

$$\cot \varrho_1 = \frac{\cos \alpha \sin \alpha}{\sin^2 \alpha} = \cot \alpha; \qquad \therefore \varrho_1 = \alpha. \quad \text{Q. E. D.}$$

14. *To prove that the sum of two radii vectores, drawn from the two foci to a common point in the curve, is equal to* 2α. — Let ϱ and ϱ' be the two radii drawn from the two foci to the common point; and since $FF' = 2\varepsilon$, we have, by trigonometry,

$$\cos \varrho' = \cos 2\varepsilon \cos \varrho - \sin \varrho \sin 2\varepsilon \cos \varphi. \qquad (31)$$

From (30) we get $\cos^2 \varepsilon = \sec^2 \beta \cos^2 \alpha$, which, multiplied by $\sin \beta$, gives $\sin^2 \beta \cos^2 \varepsilon = \tan^2 \beta \cos^2 \alpha$; and, by substitution, (27) gives

$$\cos \varphi = \frac{\sin^2 \beta \cos^2 \varepsilon \cot \varrho - \sin \alpha \cos \alpha}{\sin \varepsilon \cos \varepsilon};$$

and by this equation (31) becomes

$$\begin{aligned}
\cos \varrho' &= \cos 2\varepsilon \cos \varrho - \frac{\sin^2 \beta \cos^2 \varepsilon \cos \varrho \sin 2\varepsilon - \sin \varrho \sin 2\varepsilon \sin \alpha \cos \alpha}{\sin \varepsilon \cos \varepsilon} \\
&= \cos 2\varepsilon \cos \varrho - 2 \sin^2 \beta \cos^2 \varepsilon \cos \varrho + \sin \varrho \sin 2\alpha \\
&= \cos \varrho (\cos 2\varepsilon - 2 \sin^2 \beta \cos^2 \varepsilon) + \sin \varrho \sin 2\alpha \\
&= \cos \varrho (2 \cos^2 \varepsilon - 1 - 2 \sin^2 \beta \cos^2 \varepsilon) + \sin \varrho \sin 2\alpha \\
&= \cos \varrho (2 \cos^2 \varepsilon \cos^2 \beta - 1) + \sin \varrho \sin 2\alpha.
\end{aligned}$$

From (30) we get $\cos^2 \alpha = \cos^2 \varepsilon \cos^2 \beta$, so that the above becomes

$$\begin{aligned}
\cos \varrho' &= \cos \varrho (2 \cos^2 \alpha - 1) + \sin \varrho \sin 2\alpha \\
&= \cos \varrho \cos 2\alpha + \sin \varrho \sin 2\alpha \\
&= \cos (\varrho - 2\alpha) = \cos (2\alpha - \varrho);
\end{aligned}$$

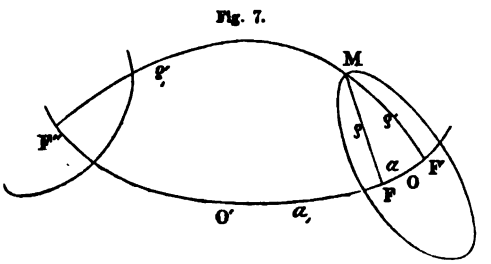
$$\therefore \varrho' = 2\alpha - \varrho, \qquad \text{or,} \qquad \varrho' + \varrho = 2\alpha. \quad \text{Q. E. D.}$$

15. It is impossible to obtain directly the equation of the spherical hyperbola, referred to either focus; or the equation of the parabola referred to its focus, as has been done in the case of the ellipse, for

in either case any plane tangent to the sphere at a focus cuts the cone only in an ellipse. We can, however, show that some of the properties of the curves already shown to form the two branches of a spherical hyperbola are analogous to those of the plane hyperbola.

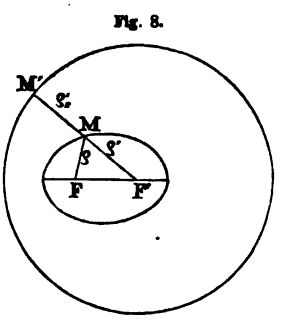
Let F and F' (Fig. 7) be the foci of one spherical ellipse, and F'' a focus in the other, distant from F' by the arc $F''FF' = F''MF' = \pi$.

Fig. 7.



Then F and F'' are the foci of the spherical hyperbola, and ρ_1' and ρ are the radii vectores, drawn from the foci to a common point M. It is required to prove that $\rho_1' - \rho = 2\alpha_1$. We know that $\rho_1' + \rho' = \pi$, and by the preceding section $\rho + \rho' = 2\alpha$. Hence, $\rho' = 2\alpha - \rho$, and $2\alpha - \rho + \rho_1' = \pi$, or $\rho_1' - \rho = \pi - 2\alpha = 2\alpha_1$. Moreover, if from one of the foci of a spherical ellipse as a centre, with a radius 2α, a circle be described, it will be the directrix of the curve considered as a spherical parabola, with the other focus as the focus of the parabola. For, since (Fig. 8)

Fig. 8.



$$F'M' = F'M + MM' = 2\alpha$$
$$= F'M + MF = \rho + \rho',$$

we have $$FM = MM',$$

or, $$\rho = \rho_2';$$

i. e. every point of the curve is equally distant from the focus and the directrix.

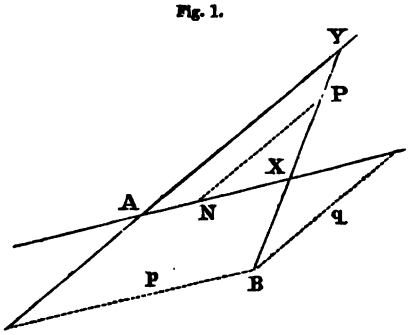
EQUATION AND CONSTRUCTION OF THE HYPERBOLA.

By JOHN WARNER, Philadelphia, Pa.

Two straight lines, AX, AY, and a point B, are given in position. Required the locus of the point P, which divides in a given ratio the intercept XY, included between the given straight lines upon any straight line drawn through B.

Take AX and AY (Fig. 1) for the axes, A for the origin at the intersection of the given straight lines; and let p and q be the co-ordinates of the point B. Let XY be always so divided that

Fig. 1.



$$XY = m\,XP.$$

Let the co-ordinates of P be $x = AN$, $y = PN$; the co-ordinates of X be $x' = AX$, $y' = 0$; and the co-ordinates of Y, $x'' = 0$, $y'' = AY$. The similar triangles, YAX, PNX, give $AX : NX :: XY : XP :: m : 1$; or, $x' : NX :: m : 1$; whence $NX = \frac{x'}{m}$, therefore $x = AN = AX - NX = x' - \frac{x'}{m}$, and $x' = \frac{m}{m-1}x$. Also, from the same similar triangles,

$$AY : PN :: XY : XP :: m : 1, \quad \text{or,} \quad y'' : y :: m : 1,$$

and $y'' = my$. Hence the preliminary equations

$$y'' = my, \qquad \text{and} \qquad x' = \frac{m}{m-1}x. \qquad (1)$$

The equation of BY which passes through B and Y, of which the co-ordinates respectively are p, q, and x'', y'', is

$$y'' - y = \frac{y'' - q}{p - x''}(x - x'').$$

Wherein $x'' = 0$ gives

$$y'' - y = \frac{y'' - q}{p} x. \tag{2}$$

And because BY passes through X and Y, of which the co-ordinates respectively are x', y', and x'', y'', its equation is

$$y'' - y = \frac{y'' - y'}{x'' - x'}(x'' - x).$$

Wherein $y' = 0$, $x'' = 0$, gives

$$y'' - y = \frac{y''}{x'} x. \tag{3}$$

By equating (2) and (3), we find $\frac{y''}{x'} = \frac{y'' - q}{p}$; whence,

$$x' y'' - p y'' - q x' = 0. \tag{4}$$

But we have, by hypothesis and by (1), $y'' = m y$, and $x' = \frac{m}{m-1} x$; hence (4) becomes, by substitution,

$$\frac{m}{m-1} xy - py - \frac{q}{m-1} x = 0, \tag{5}$$

which is the equation of the locus of P. If the origin be transposed to a point of which the co-ordinates are $x''' = \frac{m-1}{m} p$, $y''' = \frac{q}{m}$, the resulting equation will refer the curve to axes parallel to the original axes, and equation (5) will become

$$xy = (m-1)\frac{pq}{m^2}, \tag{6}$$

which is the equation of an hyperbola referred to the centre and to the asymptotes.

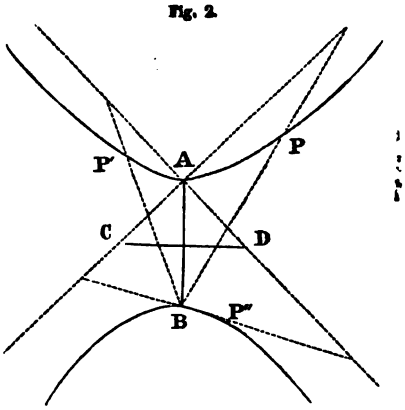
By making m variable, other curves may be generated. The foregoing investigation suggests the following construction.

To construct an hyperbola, the axes being given.

Let AB be the transverse axis, and CD the conjugate. Join one

end of the transverse, as A, with C and D, and prolong the lines AC, AD, each in both directions. From the other end of the transverse draw BP, BP', BP'', to intersect AC and AD; and bisect the intercepts included between these two lines upon the radii BP, BP', BP''. The points P, P', P'' will belong to an hyperbola, as above shown. The asymptotes will be parallel to AC and AD; but it is known that AC and AD are parallel to the asymptotes of that hyperbola of which the axes are AB and CD.

Fig. 2.



In this construction we have, evidently, $p = q = AD = AC$, and m has been taken $= 2$; hence equation (6) becomes

$$xy = \tfrac{1}{4}\overline{AC}^2 = \frac{a^2+b^2}{4},$$

which is a known property of the hyperbola, a and b being put for the semi-axes.

CONVENIENT FORMULÆ FOR INTERPOLATION.

By William Ferrel, Nautical Almanac Office, Cambridge, Mass.

Let δ^1, δ^2, δ^3, ... δ^r be the first, second, &c. orders of differences of the numbers given for interpolation, and N_0, N_1, N_2 ... N_n the numbers after interpolation, N_0 and N_m corresponding with the numbers between which the interpolations are made. Also let Δ_1, Δ_2, Δ_3 ... Δ_n be the first differences of the interpolated numbers, the

interpolations being supposed to be made at equal intervals. We have, then, the well-known equation

$$N_n = \frac{n}{m}\delta^1 + \frac{n}{m}\cdot\frac{n-m}{2m}\delta^2 + \frac{n}{m}\cdot\frac{n-m}{2m}\cdot\frac{n-2m}{3m}\delta^3 + \&c. \quad (1)$$

Put $P_n^{(e)}$ for the coefficient of δ^e in the value of N_n, then

$$\left.\begin{aligned} P_n^{(e)} &= \frac{n}{m}\cdot\frac{n-m}{2m}\cdot\frac{n-2m}{3m}\cdots\frac{n-(e-1)m}{em}, \\ P_{n-1}^{(e)} &= \frac{n-1}{m}\cdot\frac{n-1-m}{2m}\cdot\frac{n-1-2m}{3m}\cdots\frac{n-1-(e-1)m}{em}. \end{aligned}\right\} \quad (2)$$

Put
$$Q_n^{(e)} = P_n^{(e)} - P_{n-1}^{(e)}; \quad (3)$$

then
$$Q_n^{(e)} = Q_n^{(e-1)}\left(\frac{n-1-(e-1)m}{em}\right) + \frac{P_n^{(e-1)}}{em}, \quad (4)$$

and
$$\Delta_n = N_n - N_{n-1} = \Sigma_e\, Q_n^e\, \delta^e. \quad (5)$$

From equation (4) we obtain, by substituting for e the values 1, 2, 3, &c. successively, and transforming the results,

$$\left.\begin{aligned} Q_n^{(1)} &= \frac{1}{m}, \\ Q_n^{(2)} &= -\beta\cdot\frac{1}{2m^2}\cdot \\ Q_n^{(3)} &= \beta\cdot\frac{1}{4m^2} + \gamma\,\frac{1}{6m^3}, \\ Q_n^{(4)} &= -\beta\cdot\frac{4m^2-m+1}{24m^4} - \gamma\cdot\frac{1}{6m^3} + E_n^{(4)}, \\ Q_n^{(5)} &= \beta\cdot\frac{2m^2-m+1}{16m^4} + \gamma\cdot\frac{1}{6m^3} + E_n^{(5)}, \end{aligned}\right\} \quad (6)$$

in which

$$\left.\begin{aligned} \beta &= (m+1) - 2n, \\ \gamma &= \tfrac{1}{2}m^2 + \tfrac{3}{2}m + 1 + 3n^2 - 3n - 3mn, \end{aligned}\right\} \quad (7)$$

$$E_n^{(4)} = \frac{2n^3 - 3n^2 + n - 3mn^2 + 4mn + m^2n - m^2 - m}{12m^4},$$

$$E_n^{(5)} = \frac{10m^4 - 20n^3 + 20n^2 - 10n - 50mn + 120mn^2 - 120m^2n - 20m^3n - 80mn^3 - 90m^2n^2 + 5m + 30m^2 + 25m^3 - m^4}{240m^5}.$$

With these values, equation (5) gives, carrying the summation to the fifth order of differences,

$$\Delta_n = A + \beta B + \gamma C + E_n^{(4)} + E_n^{(5)}, \tag{8}$$

in which

$$\left.\begin{aligned} A &= \frac{1}{m}\delta^1, \\ B &= -\frac{1}{2m^2}\delta^2 + \frac{1}{4m^2}\delta^3 - \frac{4m^2-m+1}{24m^4}\delta^4 + \frac{2m^2-m+1}{16m^4}\delta^5, \\ C &= \frac{1}{6m^2}(\delta^3 - \delta^4 + \delta^5). \end{aligned}\right\} \tag{9}$$

The terms $E_n^{(4)}$ and $E_n^{(5)}$ are very small, and in all usual applications of the formula, in which only five orders of differences are taken in, are entirely insensible.

By substituting successively 1, 2, 3, &c. for n in the functions β and γ in the general equation (8), and neglecting the last two terms, the following special formulæ of most usual application are obtained. In most cases, the function γ for all values of n can be divided by an integer number, and C be multiplied by the same, which makes the coefficients of C in the formulæ smaller. In the last case γ is rendered divisible by 6 by deducting unity, and the effect of the same is added to the constant A. When m is an odd number, β can also be divided by 2, in which case the value of B must be doubled. In the following formulæ, small changes in some cases have been made in the fractional coefficients of δ^4 and δ^5, at a very slight expense of accuracy, in order to obtain a fraction more convenient in computation, and the maximum errors in any of the interpolated numbers arising from these changes, and the neglect of the terms $E^{(4)}$ and $E^{(5)}$, are denoted by e for those of the fourth differences, and by e' for those of the fifth differences. The values of Δ_1, Δ_2, Δ_3, &c. being obtained by the formulæ, the interpolated numbers are easily written out. A great advantage in these formulæ is, that the number of

terms used in interpolating, arising from all the orders of differences used, is much diminished; and of these, one is constant in all the values of Δ, and the others are very simple multiples of the fundamental quantities B and C, and the one half is obtained from the other half, by simply changing the signs, or reversing their order.

When $m = 3$,

$$\Delta_1 = A + 1B + 1C, \qquad \Delta_3 = A - 1B + 1C;$$
$$\Delta_2 = A - 0B - 2C,$$

in which
$$A = \tfrac{1}{3}\delta^1,$$
$$B = -\tfrac{1}{6}\delta^2 + \tfrac{1}{18}\delta^3 - \tfrac{1}{24}\delta^4 + \tfrac{1}{40}\delta^5,$$
$$C = \tfrac{1}{162}(\delta^3 - \delta^4 + \delta^5),$$
$$e = \tfrac{1}{1890}\delta^4, \qquad e' = \tfrac{1}{1000}\delta^5.$$

When $m = 4$,

$$\Delta_1 = A + 3B + 1C, \qquad \Delta_3 = A - 1B - 1C,$$
$$\Delta_2 = A + 1B - 1C, \qquad \Delta_4 = A - 3B + 1C;$$

in which
$$A = \tfrac{1}{4}\delta^1,$$
$$B = -\tfrac{1}{32}\delta^2 + \tfrac{1}{64}\delta^3 - \tfrac{1}{100}\delta^4 + \tfrac{1}{140}\delta^5,$$
$$C = \tfrac{1}{128}(\delta^3 - \delta^4 + \delta^5),$$
$$e = \tfrac{1}{1075}\delta^4, \qquad e' = \tfrac{1}{808}\delta^5.$$

When $m = 5$,

$$\Delta_1 = A + 2B + 2C, \qquad \Delta_4 = A - 1B - 1C,$$
$$\Delta_2 = A + 1B - 1C, \qquad \Delta_5 = A - 2B + 2C;$$
$$\Delta_3 = A \quad 0B - 2C,$$

in which
$$A = \tfrac{1}{5}\delta^1,$$
$$B = -\tfrac{1}{250}\delta^2 + \tfrac{1}{500}\delta^3 - \tfrac{1}{800}\delta^4 + \tfrac{1}{1100}\delta^5,$$
$$C = \tfrac{1}{2500}(\delta^3 - \delta^4 + \delta^5),$$
$$e = \tfrac{1}{16750}\delta^4, \qquad e' = \tfrac{1}{7857}\delta^5.$$

When $m = 6$,

$$\begin{aligned} \Delta_1 &= A + 5B + 5C, & \Delta_4 &= A - 1B - 4C, \\ \Delta_2 &= A + 3B - 1C, & \Delta_5 &= A - 3B - 1C, \\ \Delta_3 &= A + 1B - 4C, & \Delta_6 &= A - 5B + 5C; \end{aligned}$$

in which
$$\begin{aligned} A &= \tfrac{1}{6}\,\delta^1, \\ B &= -\tfrac{1}{72}\,\delta^2 + \tfrac{1}{144}\,\delta^3 - \tfrac{1}{225}\,\delta^4 + \tfrac{1}{3240}\,\delta^5, \\ C &= \tfrac{1}{6480}\,(\delta^3 - \delta^4 + \delta^5), \\ e &= \tfrac{1}{10800}\,\delta^4, \qquad e' = \tfrac{1}{64800}\,\delta^5. \end{aligned}$$

When $m = 8$,

$$\begin{aligned} \Delta_1 &= A + 7B + 7C, & \Delta_5 &= A - 1B - 5C, \\ \Delta_2 &= A + 5B + 1C, & \Delta_6 &= A - 3B - 3C, \\ \Delta_3 &= A + 3B - 3C, & \Delta_7 &= A - 5B + 1C, \\ \Delta_4 &= A + 1B - 5C, & \Delta_8 &= A - 7B + 7C; \end{aligned}$$

in which
$$\begin{aligned} A &= \tfrac{1}{8}\,\delta^1, \\ B &= -\tfrac{1}{128}\,\delta^2 + \tfrac{1}{256}\,\delta^3 - \tfrac{1}{400}\,\delta^4 + \tfrac{1}{4800}\,\delta^5, \\ C &= \tfrac{1}{10240}\,(\delta^3 - \delta^4 + \delta^5), \\ e &= \tfrac{1}{12250}\,\delta^4, \qquad e' = \tfrac{1}{48000}\,\delta^5. \end{aligned}$$

When $m = 10$,

$$\begin{aligned} \Delta_1 &= A + 9B + 6C, & \Delta_6 &= A - 1B - 4C, \\ \Delta_2 &= A + 7B + 2C, & \Delta_7 &= A - 3B - 3C, \\ \Delta_3 &= A + 5B - 1C, & \Delta_8 &= A - 5B - 1C, \\ \Delta_4 &= A + 3B - 3C, & \Delta_9 &= A - 7B + 2C, \\ \Delta_5 &= A + 1B - 4C, & \Delta_{10} &= A - 9B + 6C; \end{aligned}$$

in which
$$\begin{aligned} A &= \tfrac{1}{8}\,\delta^1, \\ B &= -\tfrac{1}{2000}\,\delta^2 + \tfrac{1}{4000}\,\delta^3 - \tfrac{1}{6144}\,\delta^4 + \tfrac{1}{8400}\,\delta^5, \\ C &= \tfrac{1}{10000}\,(\delta^3 - \delta^4 + \delta^5), \\ e &= \tfrac{1}{6000}\,\delta^4, \qquad e' = \tfrac{1}{8000}\,\delta^5, \end{aligned}$$

When $m = 12$,

$$\begin{aligned}
\Delta_1 &= A + 11B + 9C, & \Delta_7 &= A - 1B - 6C,\\
\Delta_2 &= A + 9B + 4C, & \Delta_8 &= A - 3B - 5C,\\
\Delta_3 &= A + 7B + 0C, & \Delta_9 &= A - 5B - 3C,\\
\Delta_4 &= A + 5B - 3C, & \Delta_{10} &= A - 7B + 0C,\\
\Delta_5 &= A + 3B - 5C, & \Delta_{11} &= A - 9B + 4C,\\
\Delta_6 &= A + 1B - 6C, & \Delta_{12} &= A - 11B + 9C;
\end{aligned}$$

in which $A = \frac{1}{12}\delta^1 + \frac{1}{6}C$,

$$B = -\frac{1}{288}\delta^2 + \frac{1}{576}\delta^3 - \frac{1}{880}\delta^4 + \frac{1}{1200}\delta^5,$$

$$C = \frac{1}{1728}(\delta^3 - \delta^4 + \delta^5),$$

$$e = \frac{1}{576}\delta^4, \qquad e' = \frac{1}{800}\delta^5.$$

The following is an example of the application of the last formula to the interpolation of the moon's right-ascension from half-days to hours. It is taken where the fourth and fifth differences are unusually large.

R. A.	δ^1	δ^2	δ^3	δ^4	δ^5	A.	B.	C.
h m s	m s	s	s	s	s	m s	s	s
2 12 47.60	+29 24.67	+19.59	−5.17	−1.81	+0.52	+2 27.056	−0.0753	−0.0016
2 42 12.27	29 44.26	14.42	6.98	1.29	0.69	2 28.688	0.0602	0.0028
3 11 56.53	29 58.68	7.44	8.27	0.60	0.80	2 29.890	0.0386	0.0036
. . . . .	. . . . .	. . .	. . .	. . .	. . .	. . . . .	. . . . .	. . . . .

In the first of the following columns are written the functions $11B$, $9B$, &c., and in the second, $9C$, $4C$, &c. The sum of these is placed in the third column, to which the constant A is added for Δ^1, or the fourth column, the numbers of which are the first differences of the interpolated numbers. The differences of Δ^1, i. e. Δ^2, should be taken as a test of the accuracy of the work before writing out the interpolated numbers.

Great care should be taken to get the quantities A, B, and C very accurately, and when a series of interpolations are to be made, they should be tested by means of the differences, before proceeding to

interpolate, in order to save the trouble of correcting whole groups of numbers, when the errors are discovered by the subsequent tests.

Functions of B.	Functions of C.	Sum of Functions.	Δ^1	Δ^2	R. A.
			m s		h m s
—.828	—.014	—.842	2 26.214	.158	2 12 47.60
.678	.006	.684	.372	.157	15 13.81
.527	—.000	.527	.529	.156	17 40.18
.376	+.005	.371	.685	.153	20 6.71
.226	.008	.218	.838	.153	22 33.39
—.075	.010	—.065	.991	.150	25 0.23
+.075	.010	+.085	27.141	.149	27 27.22
.226	.008	.234	.290	.147	29 54.36
.376	.005	.381	.437	.146	32 21.65
.527	+.000	.527	.583	.145	34 49.09
.678	—.006	.672	.728	.142	37 16.67
+.828	.014	+.814	.870	.131	39 44.40
—.662	.025	—.687	28.001	.134	42 12.27
.542	.011	.553	.135	.132	44 40.27
.421	—.000	.421	.267	.128	47 8.40
.301	+.008	.293	.395	.126	49 36.67
.181	.014	.167	.521	.124	52 5.07
—.060	.017	—.043	.645	.120	54 33.59
+.060	.017	+.077	.745	.118	57 2.23
.181	.014	.195	.883	.114	2 59 31.00
.301	.008	.309	.997	.112	3 1 59.88
.421	+.000	.421	29.109	.110	4 28.88
.542	—.011	.531	.219	.106	6 57.99
+.662	—.025	+.637	.325		9 27.21
...	...	...	...	...	3 11 56.53
					. .. .

The maximum error, in this example of the moon's right ascension, from neglecting the terms E^4 and E^5, is only about $0^s.003$, and is generally less than $0^s.002$.

If in equation (8) we substitute for β, γ, A, B, and C, their values in equations (7) and (9), and suppose m to be infinitely great,

Δ_n becomes dN, and $\frac{1}{m}$ becomes $d\frac{n}{m}$, and neglecting terms infinitely small, it becomes

$$m D_n N = \tfrac{1}{2} C' \frac{n^2}{m^2} - (2B' + \tfrac{1}{2} C') \frac{n}{m} + A' + B' + \tfrac{1}{12} C, \quad (10)$$

in which

$$\left.\begin{aligned} A' &= \delta', \\ B' &= -\tfrac{1}{2}\delta^2 + \tfrac{1}{4}\delta^3 - \tfrac{1}{6}\delta^4 + \tfrac{1}{8}\delta^5, \\ C' &= \delta^3 - \delta^4 + \delta^5. \end{aligned}\right\} \quad (11)$$

Putting $D_n N = 0$, we get

$$\frac{n^2}{m^2} - \left(\frac{4B'}{C'} + 1\right)\frac{n}{m} = -2\,\frac{(A' + B')}{C'} - \tfrac{1}{6}. \quad (12)$$

The value of $\frac{n}{m}$ obtained from this equation is the fraction of the interval at which N becomes a maximum or a minimum. By means of this equation the exact place of the moon's perigee or apogee, or of a planet's perihelion or aphelion, the moon's greatest libration, &c., can be very conveniently obtained without interpolating to short intervals.

Integrating equation (10), we get

$$N - N' = \tfrac{1}{6} C' \frac{n^3}{m^3} - (B' + \tfrac{1}{4} C') \frac{n^2}{m^2} + (A' + B' + \tfrac{1}{12} C') \frac{n}{m}, \quad (13)$$

N' being the value of N when $n = 0$. By means of this equation, the exact place or fraction of interval can be determined at which N becomes of a given value, and it can be used in determining the phases of the moon, and in all similar cases.

ALPHABETICAL INDEX.

Page

Algebra, Todhunter's, Notes from, 52, 53, 54
——— for the Use of Colleges and Schools, by Todhunter, Notice of, . . . 62
——— University, by Davies, Notice of, 63
Algebraic Curves, A General Method of finding Tangents to, by Holliday, . 149
Analysis, On the Diophantine, by A. D. Wheeler, ... 325
——— Spherical, by Eastwood, . 78
Arithmetic, A Logical Outline of, by Brooks, ... 182
——— Mean of any Number of Positive Quantities is greater than the Geometric Mean, ... 69
Astronomical Notices, edited by Brünnow, Notice of, ... 64
Astronomy, A Treatise on, by Robinson, Notice of, ... 125
Asteroids, On the Secular Variations and Mutual Relations of the Orbits of, by Newcomb, Notice of, ... 95
Bardwell, Prof. F. W., Note on the Horizontal Thrust of Embankments, . 6
Barrow, Isaac, D. D., The Mathematical Works of, Notice of, ... 286
Binomial Theorem, Demonstration of, . 54
Binomials, Note on Powers of, ... 69
Borden, John, Note on the Dependence of Equations, ... 7
Brooks, Prof. Edward, A Logical Outline of Arithmetic, ... 182
Calculus, Differential and Integral, by Church, Notice of, ... 289

Page

Calculus, Differential and Integral, by Davies, Notice of, ... 30
——— by Smyth, Notice of, . . . 159
Calculus of Variations, A History of the Progress of, during the Nineteenth Century, by Todhunter, Notice of, . . . 354
Chart and Diagram for facilitating Great-Circle Sailing, by Godfray, Notice of, 94
Chemical Physics, Elements of, by Cooke, Notice of, ... 30
Church, Prof. A. E., Elements of the Differential and Integral Calculus, Notice of, ... 289
Circle, On an Approximate and Graphical Rectification of, by Herschel, . . 153
——— On the Quadrature of, by Smith, Notice of, ... 291
Circular Arc, Time of Vibration in, by Gibbes, ... 40
Collins, Matthew, Mathematical Notes, 306
Common Measure, Note on the Greatest, by Hunt, ... 51
Conics, Spherical, by Woodward, . 361
Cooke, Josiah P., Jr., Elements of Chemical Physics, Notice of, ... 30
Cubes of the first n Natural Numbers, Sum of the, ... 53
Curves, On the Geometrical Construction of, by Points, by Newton, . . 235, 268
Davies, Prof. Charles, Elements of Analytical Geometry and Differential and Integral Calculus, Notice of, . . . 30
——— University Algebra, Notice of, 63

Determinants, A Treatise on, by OLIVER, 86
Diary, The Lady's and Gentleman's for 1861, Notice of, ... 159
——— Problems from, ... 190
Differential Calculus, The Claim of LEIBNITZ to the Invention of, by SLOMAN, Notice of, ... 239
DODD, JAMES B., Series of Mathematical Text-Books, Notice of, ... 290
Earth, On the Conformation of the, Prize Essay, by HILL, ... 166
EASTWOOD, GEORGE, On Spherical Analysis, ... 78
Editorial Items, . . 64, 160, 192, 228, 292
ELLIS, THEODORE G., The Theory of the Gyroscope, ... 209
Embankments, Horizontal Thrust of, by BARDWELL, ... 6
Equations, Note on the General Theory of, by EVANS, ... 256
——— Note on Numerical, by VALLAS, 10
——— Numerical, of the Third Degree, Solution of, by PORTER, . . . 198
——— On the Dependence of, by BORDEN, ... 7
——— which determine the Position of a Comet or other Planetary Body, from Three Observations, Discussion of, by HILL, ... 26
Equilibrium, Demonstration of the Law of, in the Lever, by STRONG, . . . 283
EVANS, ASHER B., Solution of Problem V., No. X., Vol. II., ... 151
——— Prof. E. W., Note on the General Theory of Equations, ... 256
FERREL, WILLIAM, Convenient Formulæ for Interpolation, ... 377
Fraction, Continued, To change a Series into, by HALL, ... 262
Geodetic Formulas, by MATTHEWSON, 71
Geometry, Note on, by POWERS, . . 185
——— Analytical, Note on, by STEVENS, 185
——— Analytical, by DAVIES, Notice of, 30
——— Descriptive, General Problems from the Orthographic Projections of, by WARREN, Notice of, . . . 155
Geometry, Descriptive, of One Plane, by WATSON, ... 133
——— Elements of, by GREENLEAF, Notice of, ... 32
GIBBES, Prof. LEWIS R., An Easy Mode of Approximating to the Time of Vibration in a Circular Arc, ... 40
GODFRAY, HUGH, Note on the Problem of the Lights, ... 38
Great-Circle Sailing, or a Chart and Diagram for facilitating, by GODFRAY, Notice of, ... 94
GREENLEAF, BENJAMIN, Elements of Geometry, Notice of, ... 32
Gyroscope, Theory of, by ELLIS, . . 209
HALL, A., To Change a Series into a Continued Fraction, ... 262
HENKLE, Prof. W. D., Note on Perfect Squares, ... 66
HERSCHEL, A. S., On an Approximate and Graphical Rectification of the Circle, 152
——— Sir JOHN, A Complete Catalogue of the Writings of, ... 220
HILL, G. W., Discussion of the Equations which determine the Position of a Comet or other Planetary Body, from Three Observations, ... 26
——— Prize Essay, On the Conformation of the Earth, ... 166
HOYT, Prof. D. W., Note on some Properties of Square Numbers, . . . 12
——— Note on Prize Solution of Problem III., p. 25, ... 67
HOLLIDAY, WALLER, A General Method of finding Tangents to Algebraic Curves, 149
HUNT, E., Note on the Greatest Common Measure, ... 51
Hyperbola, Equation and Construction of, by WARNER, ... 375
Inclined Plane, Motion on, by SHERWIN, 206
Insurance, by NEWCOMB, ... 343
Interest for Days, Correction of the Methods for Computing Six Per Cent, . . 67
——— Note on a Question in, by PORTER, 255
Interpolation, Convenient Formulæ for, by FERREL, ... 377

Items, Editorial, . 64, 160, 192, 228, 292
Leibnitz, The Claim of, to the Invention of the Differential Calculus, by Sloman, Notice of, ... 259
Lever, Demonstration of the Law of Equilibrium in, by Strong, ... 283
Magic Table, by Martin, ... 189
Mathematical Monthly Notices, 30, 61, 90, 125, 155, 190, 259, 286, 354
Mathematical Development, Exposition of the Process of, by Paterson, . . 55, 108
Mathematical Notes, by Collins, . . 306
Mathematical Operations, by Robinson, Notice of, ... 90
Mathematical Works of Isaac Barrow, Notice of, ... 286
Matthewson, Dr. R. C., The Earth considered as a Spheroid of Revolution, . 71
Morgan, H. A., A Collection of Problems and Examples, Notice of, . . . 64
Napier's Rules of Circular Parts, Note on, 99
Natural Philosophy, Introductory Course of, by Peck, Notice of, ... 128
Newcomb, S., Solution of Problem V., No. IX., Vol. II., ... 68
——— Notes on Probabilities, . 119, 343
——— On the Secular Variations and Mutual Relations of the Orbits of the Asteroids, Notice of, ... 95
Newton, Prof. H. A., On the Geometrical Construction of certain Curves by Points, ... 335, 268
Note on a Problem in Robinson's Algebra, by Hoyt, ... 189
Note on Successive Derivatives of tan φ, . 70
Notes and Queries, 51, 66, 98, 149, 182, 254
Notes on Probabilities, by Newcomb, 119, 343
Numerical Equations of the Third Degree, Geometrical Method of Solution of, by Porter, ... 203
Oliver, J. E., A Treatise on Determinants, ... 86
Parker, Prof. W. H., Remarks on the Problem of the Lights, ... 34
Paterson, John, Exposition of the Process of Mathematical Development, 55, 108
Paterson, John, On the Results of Accelerated Velocity, ... 357
Peck, Prof. W. G., Introductory Course of Natural Philosophy, Notice of, . . 128
Pendulum, by Sherwin, ... 312
Perfect Squares, Note on, by Henkle, . 66
Polytechnic School at Paris, by Watson, ... 320
Porter, Prof. J. C., Note on a Question in Interest, ... 254
——— Solution of Numerical Equations of the Third Degree, ... 198
Prize Essay, On Central and Disturbing Forces, by Trowbridge, . . . 245
——— On the Conformation of the Earth, by Hill, ... 166
——— Spherical Conics, by Woodward, 361
——— The Method of Projections, by Wright, ... 293
Prize Essays, Report of Committee on, . 160
Prize Problems for Students, 1, 33, 65, 97, 129, 161, 193, 229, 261
——— Solutions of, 18, 46, 104, 130, 162, 194, 230, 280, 349
Prize Solution of Problem III., p. 25, Note on, by Hoyt, ... 67
Probabilities, Notes on, by Newcomb, 119, 343
Problem of the Lights, Note on, by Godfray, ... 38
——— Remarks on, by Parker, . 34
Problem, Solution of, by Trowbridge, . . 187
Problems and Examples, A Collection of, by Morgan, Notice of, . . . 64
Problems and Riders for the Year 1860, Cambridge Senate-House, with Solutions of, by Watson and Routh, Notice of, 61
Projections, The Method of, by Wright, . 293
Quadrature of the Circle, by Smith, Notice of, ... 291
Recurring Decimals, Values of, . . 53
Roberson, J. F., Square Numbers ending in 5, 25, 75, ... 150
Robinson, H. N., A Treatise on Astronomy, Notice of, ... 125
——— Concise Mathematical Operations, Notice of, ... 90